cem
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POEM: Physics of Emergent Materials

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L1: Spin Orbit Coupling

L2: Topology and Topological Insulators

Tutorials: May 24, 25 (2017)

Scope of Lectures and Anchor Points:

1. Spin-Orbit Interaction
- atomic SOC
- band SOC: dresselhaus and rashba
- symmetries: time reversal, inversion, mirror

2. Berry Phase and Topological Invariant
- two level system
- graphene

3. Hall effects
- integer qhe and chern #

Recap

?

$$\vec{d}(\vec{k})$$

NO SOC :

$$\mathcal{H} = \sum_{\vec{k}\sigma} \epsilon(\vec{k})\, c^{+}_{k\sigma}\, c_{k\sigma}$$

$$= \sum_{\vec{k}} \begin{pmatrix} c^{+}_{k\uparrow} & c^{+}_{k\downarrow} \end{pmatrix} \begin{pmatrix} \epsilon(\vec{k}) & 0 \\ 0 & \epsilon(\vec{k}) \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix}$$

SOC ≠ 0

$$\mathcal{H} = \sum_{\vec{k}} \begin{pmatrix} c^{+}_{k\uparrow} & c^{+}_{k\downarrow} \end{pmatrix} \begin{pmatrix} h_{\uparrow\uparrow}(\vec{k}) & h_{\uparrow\downarrow}(\vec{k}) \\ h_{\downarrow\uparrow}(\vec{k}) & h_{\downarrow\downarrow}(\vec{k}) \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix}$$

SOC ≠ 0

$$\mathcal{H} = \sum_{\vec{k}} \begin{pmatrix} c_{k\uparrow}^{+} & c_{k\downarrow}^{+} \end{pmatrix} \begin{pmatrix} h_{\uparrow\uparrow}(\vec{k}) & h_{\uparrow\downarrow}(\vec{k}) \\ h_{\downarrow\uparrow}(\vec{k}) & h_{\downarrow\downarrow}(\vec{k}) \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix}$$

Effect of SOC :

In real space as election hops from one site to another its spin can flip.

Using Pauli matrices

$$\vec{h}(\vec{k}) = \epsilon(\vec{k})\, I + \vec{d}(\vec{k}) \cdot \vec{\sigma}$$

$$= \epsilon(\vec{k}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + d_x(\vec{k})\,\sigma_x + d_y(\vec{k})\,\sigma_y + d_z(\vec{k})\,\sigma_z$$

(suppress $\vec{k}$

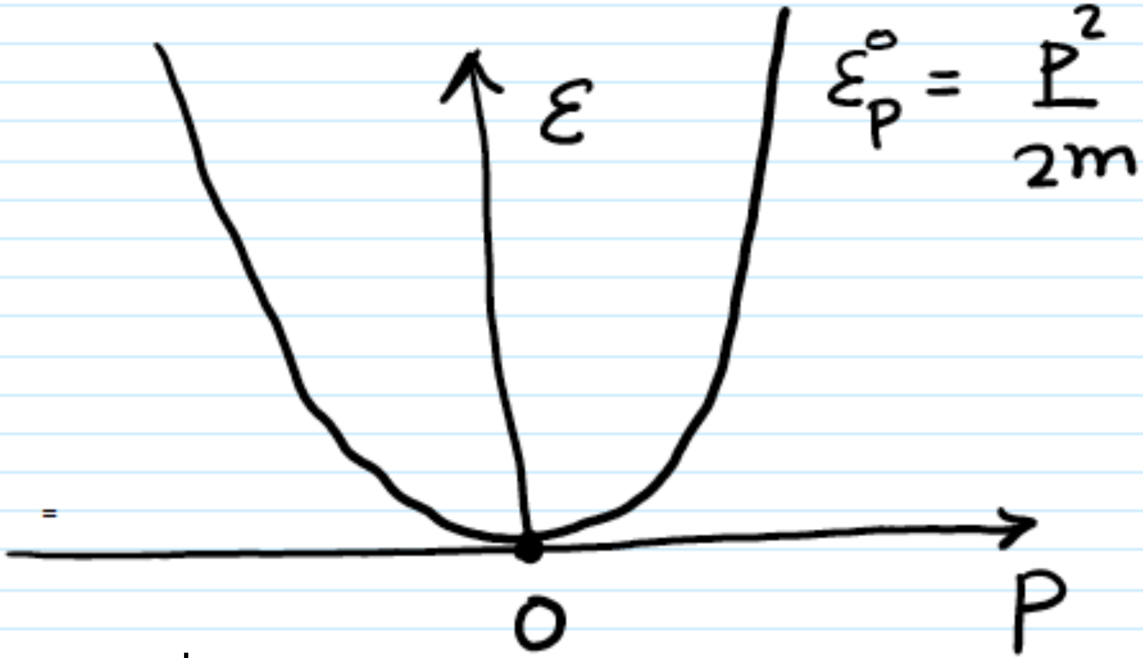
$$= \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + d_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + d_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \epsilon + d_z & d_x - i d_y \\ d_x + i d_y & \epsilon - d_z \end{pmatrix}$$

Spin – momentum coupling

$$\mathcal{H} = \frac{p^2}{2m}\mathbb{1} - \lambda\,\sigma_z\,p$$

$\lambda = 0$



Each eigenstate is doubly degenerate:
Kramer's degeneracy

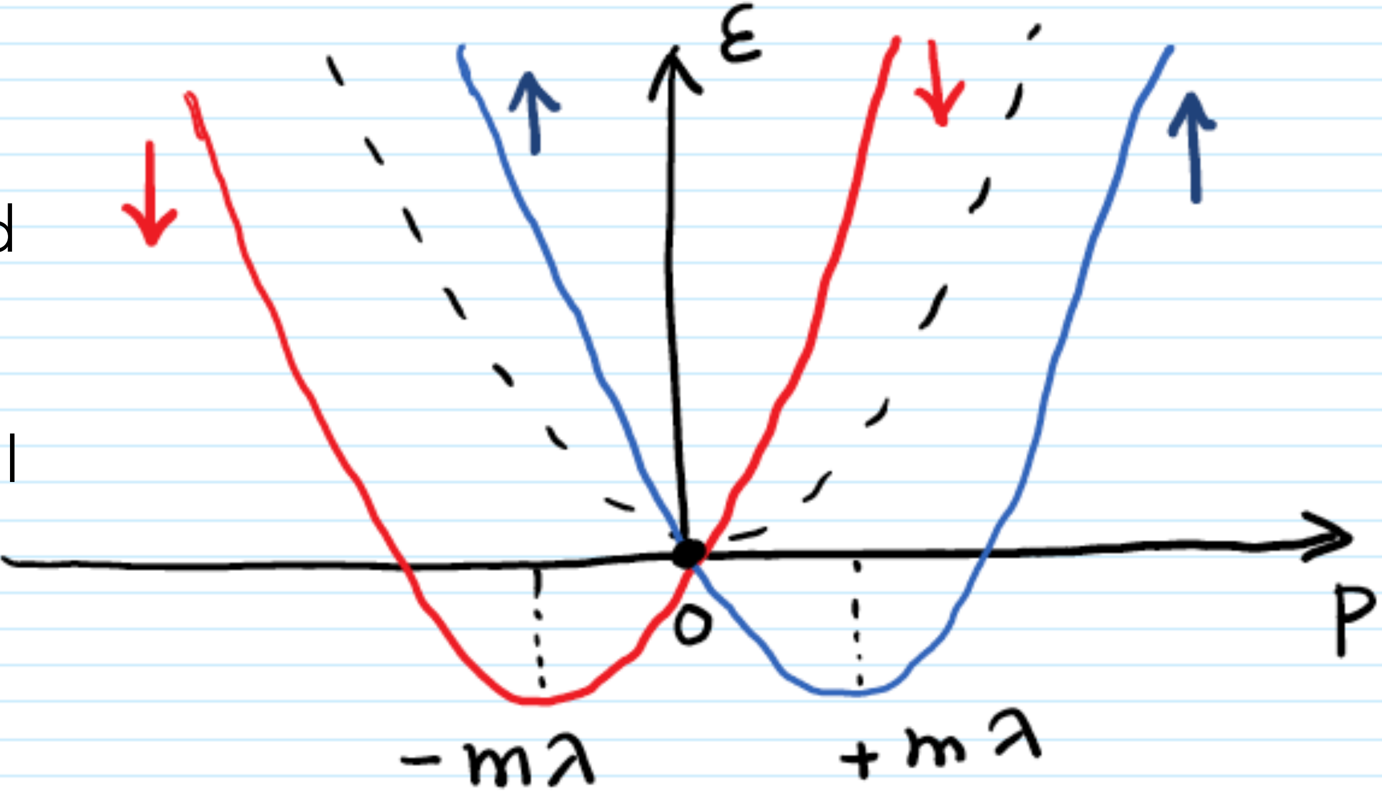
$\lambda \neq 0$

$$\mathcal{H} = \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{pmatrix} \overset{\uparrow}{\varepsilon_p^0 - \lambda p} & \overset{\downarrow}{0} \\ 0 & \varepsilon_p^0 + \lambda p \end{pmatrix}$$

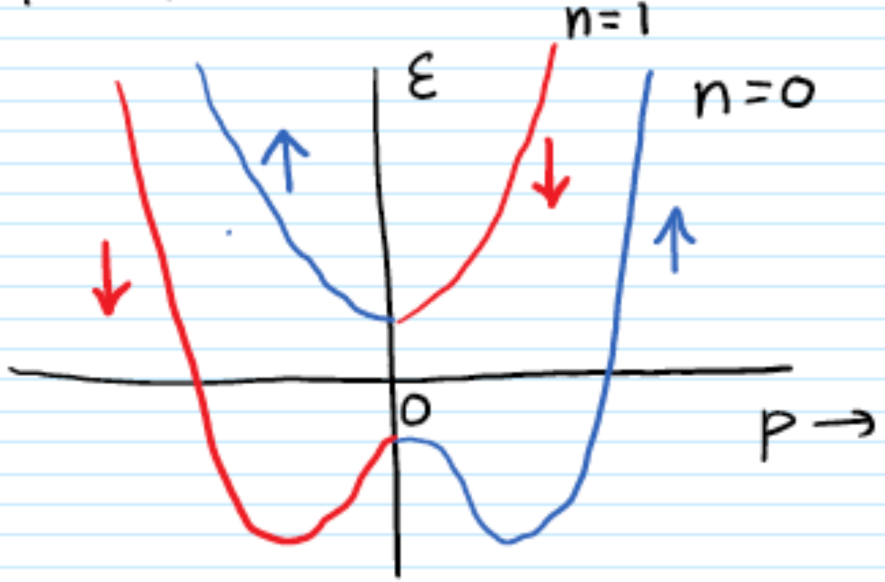
Degeneracy between up and down spins is lifted by SOC

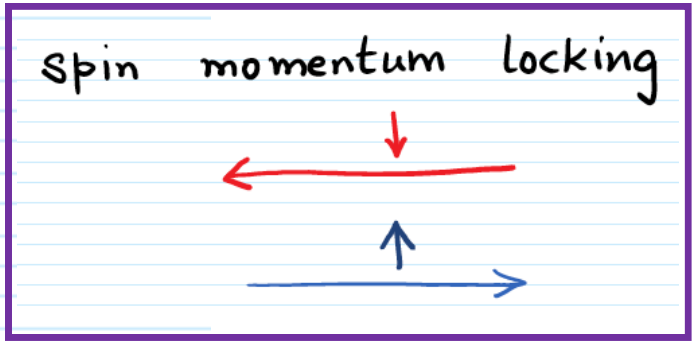
Except at specific TRIM: time reversal invariant momenta

e.g. p=0



Apply small Zeeman field along x





Gap opens up at the degenerate point.

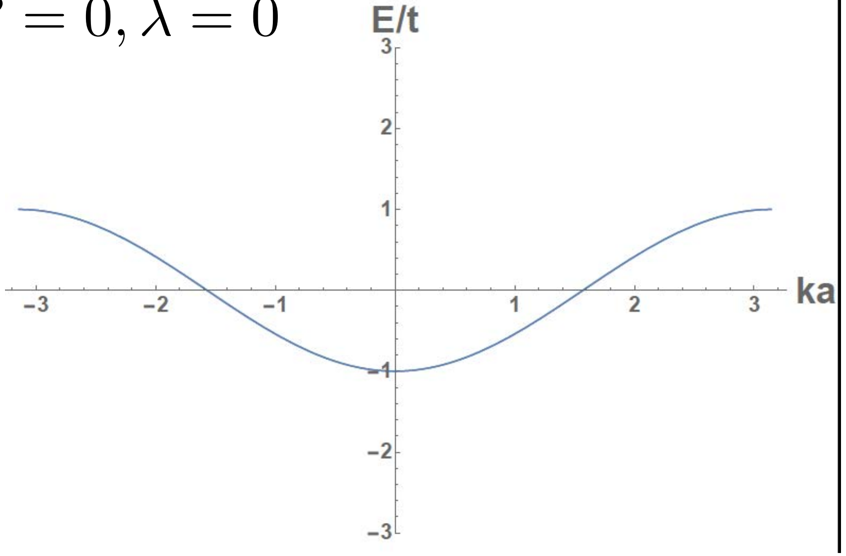
Traverse the BZ always remaining in the lowest band. (adiabatic evolution).

→ The electron spin will twist from ↓ for $-p$ to ↑ for $+p$

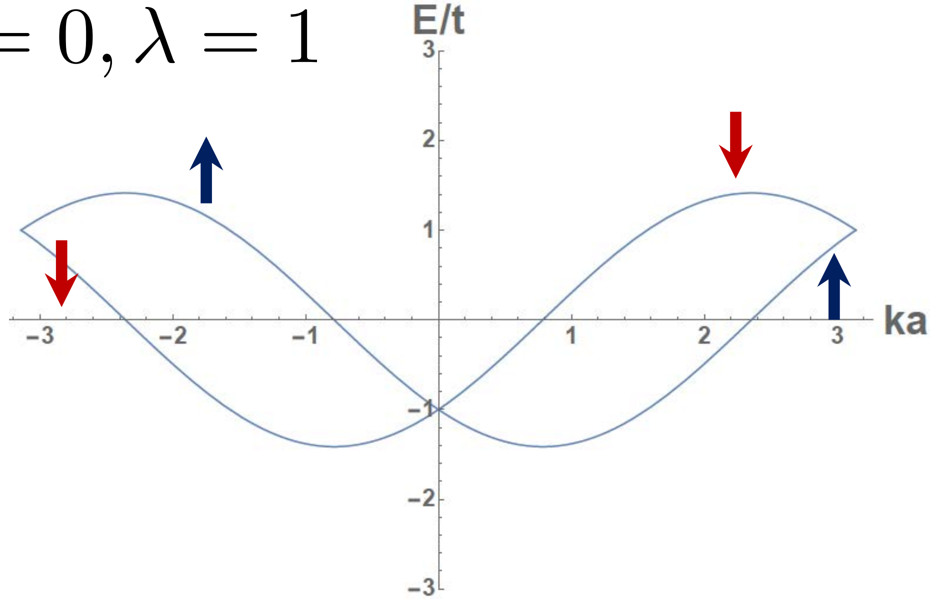
$$h(k) = -\cos(k)\sigma_0 - \lambda\sin(k)\sigma_z - B\sigma_x = -\cos(k)\sigma_0 + \vec{d}(k).\vec{\sigma}$$

$$d_x(k) = -B \quad d_z(k) = -\lambda\sin(k)$$

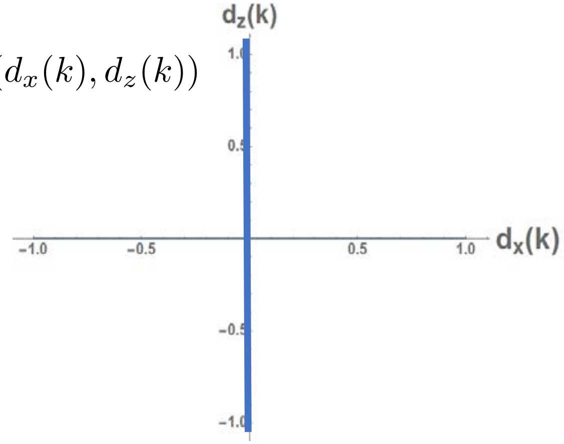
$B = 0, \lambda = 0$



$B = 0, \lambda = 1$

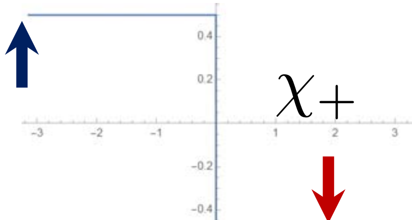


$(d_x(k), d_z(k))$



$\chi-$

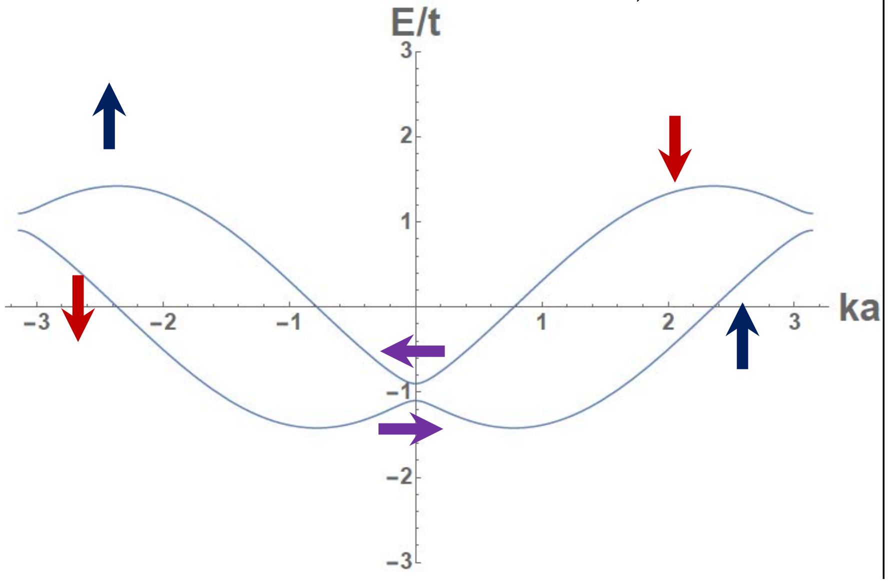
Eigenvector corresponding to lower energy

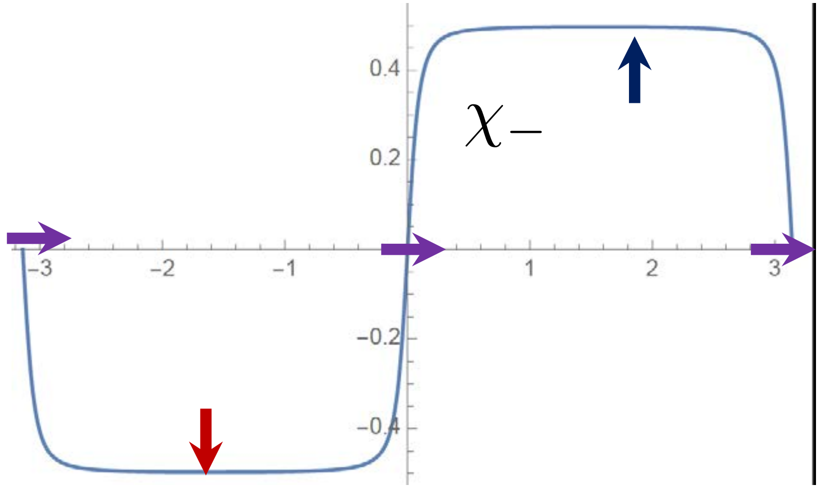


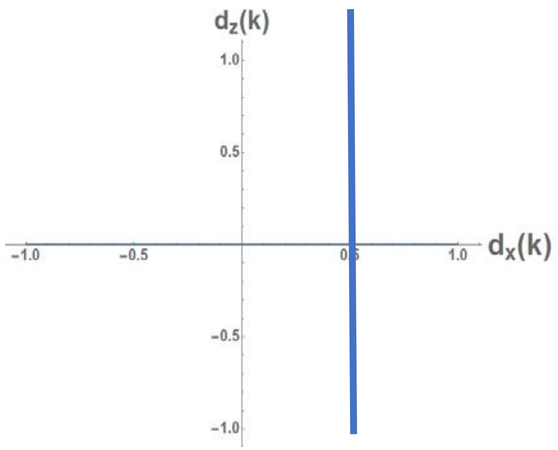
$\chi+$

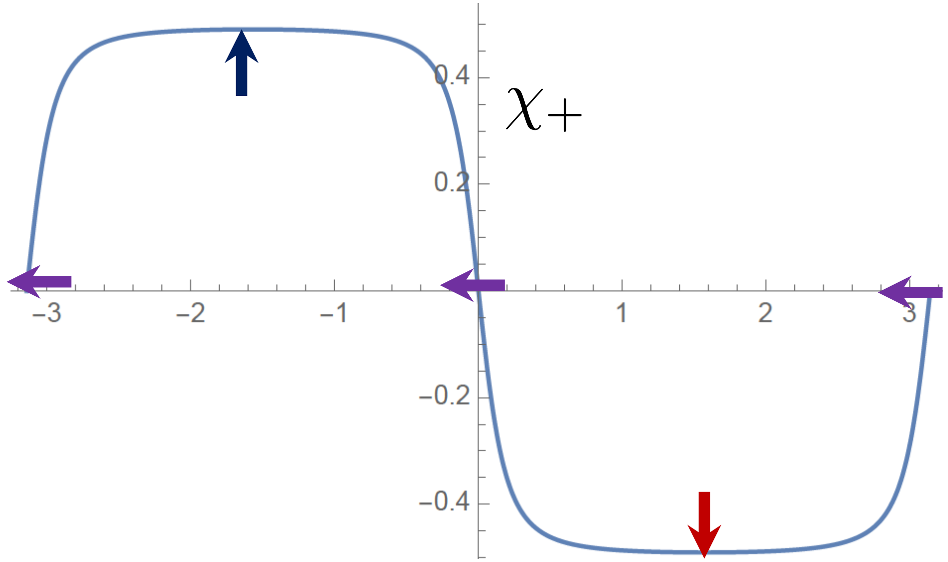
Eigenvector corresponding to upper energy

$B = 0.1, \lambda = 1$







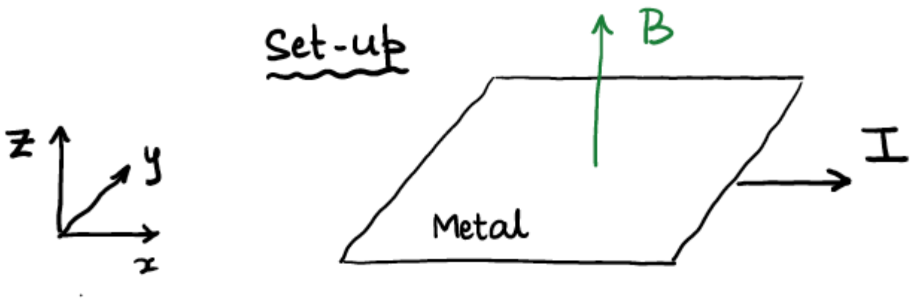


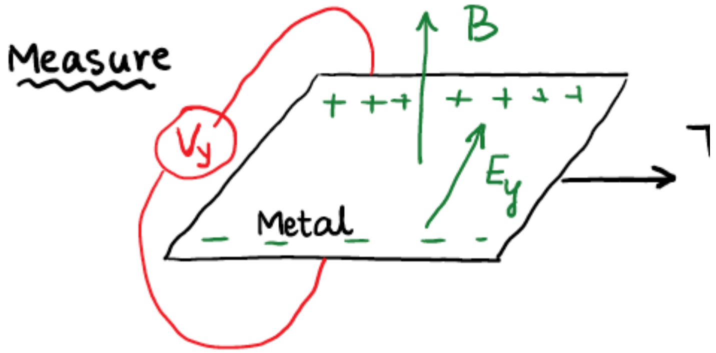
No Winding: hence topologically trivial

Why is all this important?

Interesting?
Useful?
Fundamentally important?
Game changer?
Can we get a device out of this?
…much more than a device!
A whole new paradigm for information storage that is topologically protected!

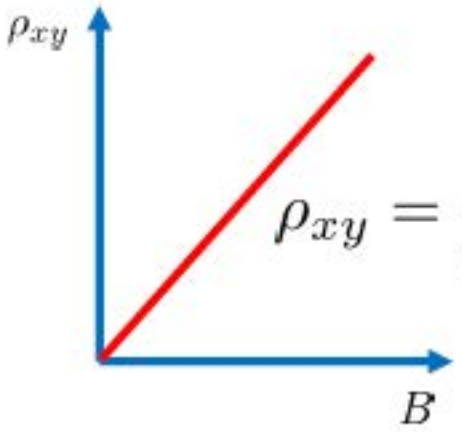
Hall Effect



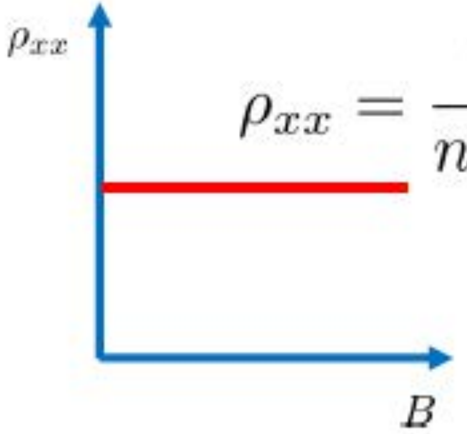


$$eE_y = ev_x B$$

$$j_x = nev_x$$



$$\rho_{xy} = \frac{B}{nec}$$



$$\rho_{xx} = \frac{m}{ne^2\tau}$$

$$\rho_{xx} = \frac{E_x}{j_x}$$

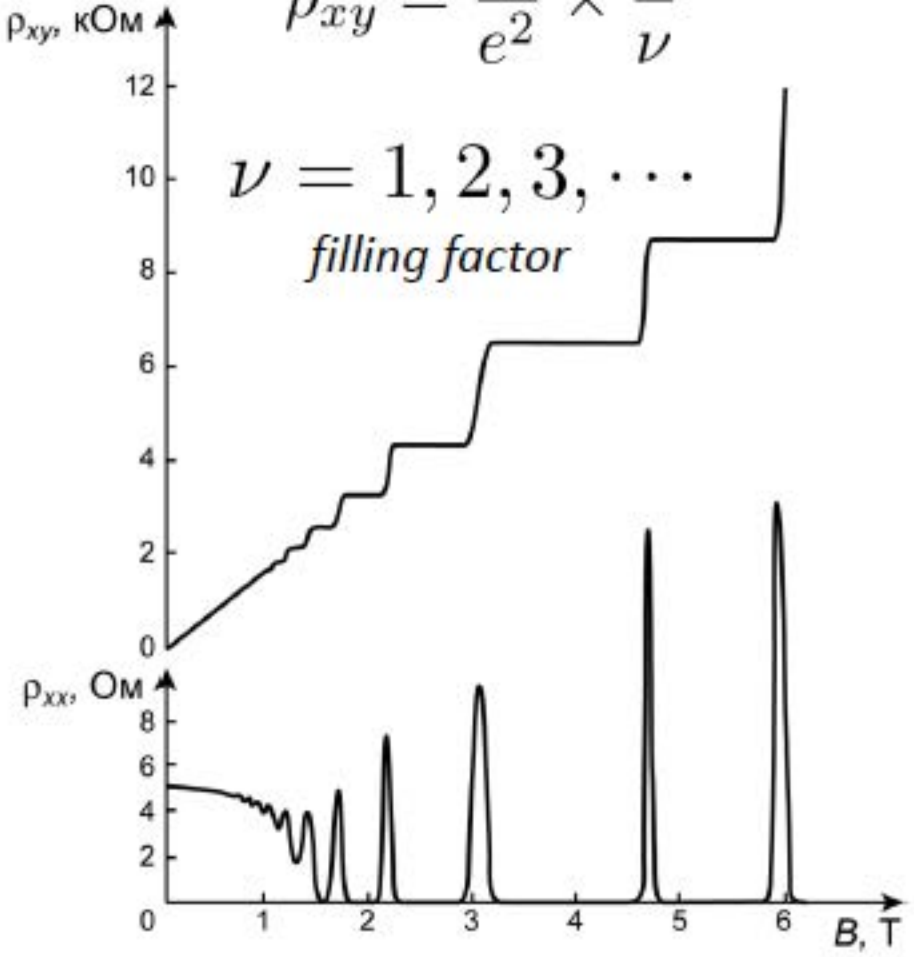
$$\rho_{xy} = \frac{E_y}{j_x}$$

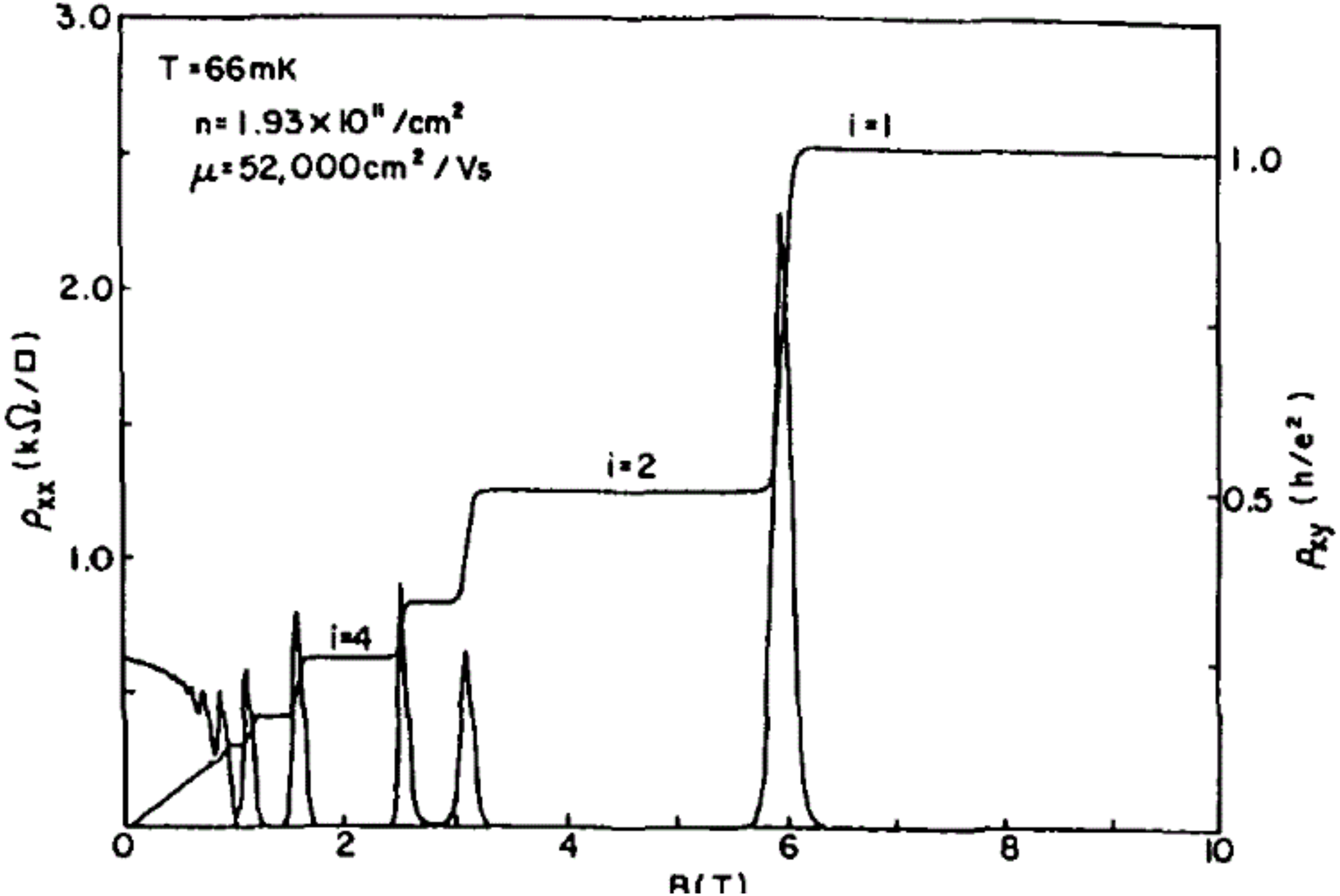
Quantum Hall Effect

$$\rho_{xy} = \frac{h}{e^2} \times \frac{1}{\nu}$$

$$\nu = 1, 2, 3, \cdots$$

filling factor





$$\overleftrightarrow{\sigma} = [\overleftrightarrow{\rho}]^{-1}$$

$$\sigma_{xx} = 0$$

$$\sigma_{xy} = \frac{e^2}{h}n$$

$$= \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2}$$

$$\sigma_{xx} \propto \rho_{xx}$$

$$= \frac{\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2}$$

$$\sigma_{xy} = \frac{1}{\rho_{xy}}$$

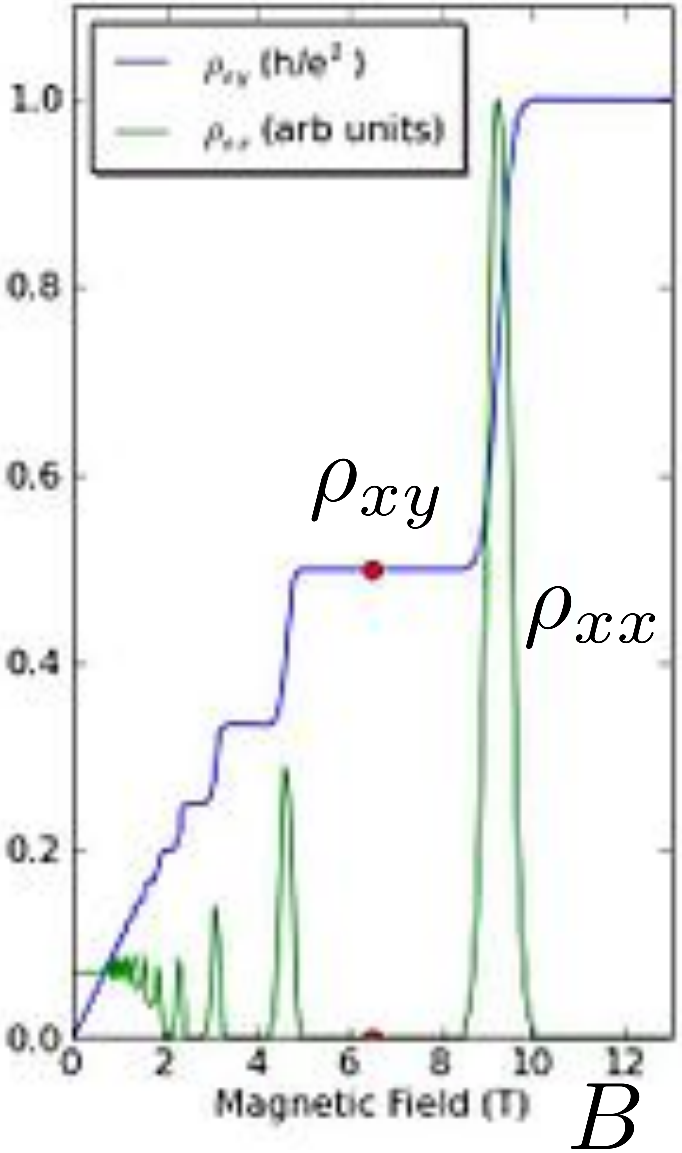
Topology: Shapes and invariants

quantum hall effect

precise quantization of Hall conductance to *one part in billion* even though system is disordered

$$\sigma_{xy} = \frac{e^2}{h} n$$ Chern number

SOC is a source of "magnetic field" in momentum space k
What kinds of Hall effects can it produce?
What kinds of invariants does it lead to?



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How large is the effective magnetic field arising from SOC?

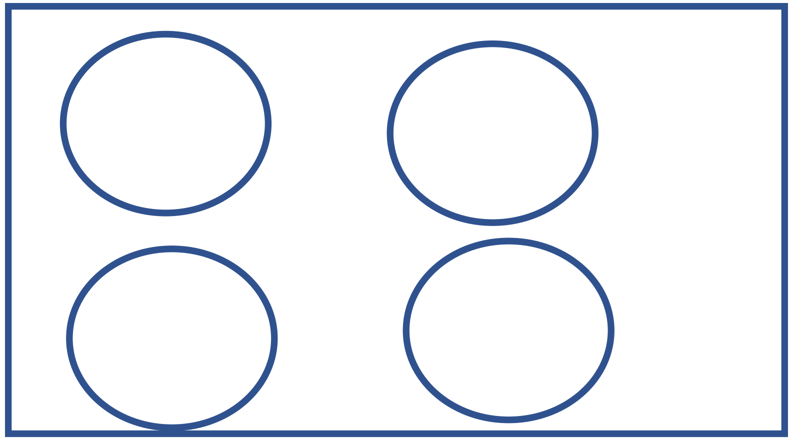
$$\mu_B \approx 10^{-4} \ \frac{\mathrm{eV}}{\mathrm{Tesla}}$$

$$\lambda_{\mathrm{soc}} \approx 0.1 eV$$

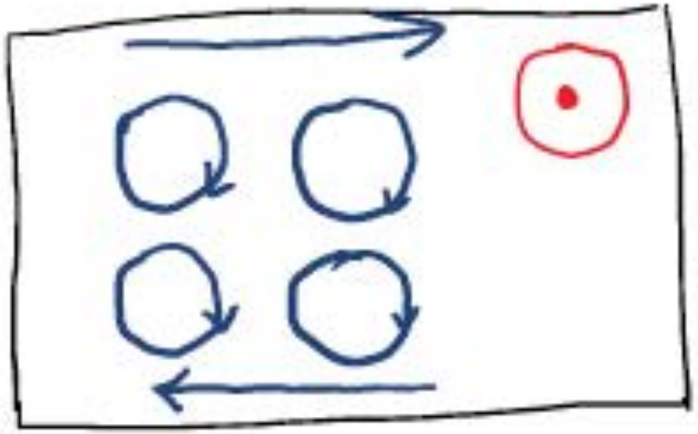
$$\Rightarrow b_{\mathrm{soc}}^{\mathrm{eff}} \approx 1000 \mathrm{Tesla}$$

insulators

Ordinary insulator
(atomic or band)



Hall insulator



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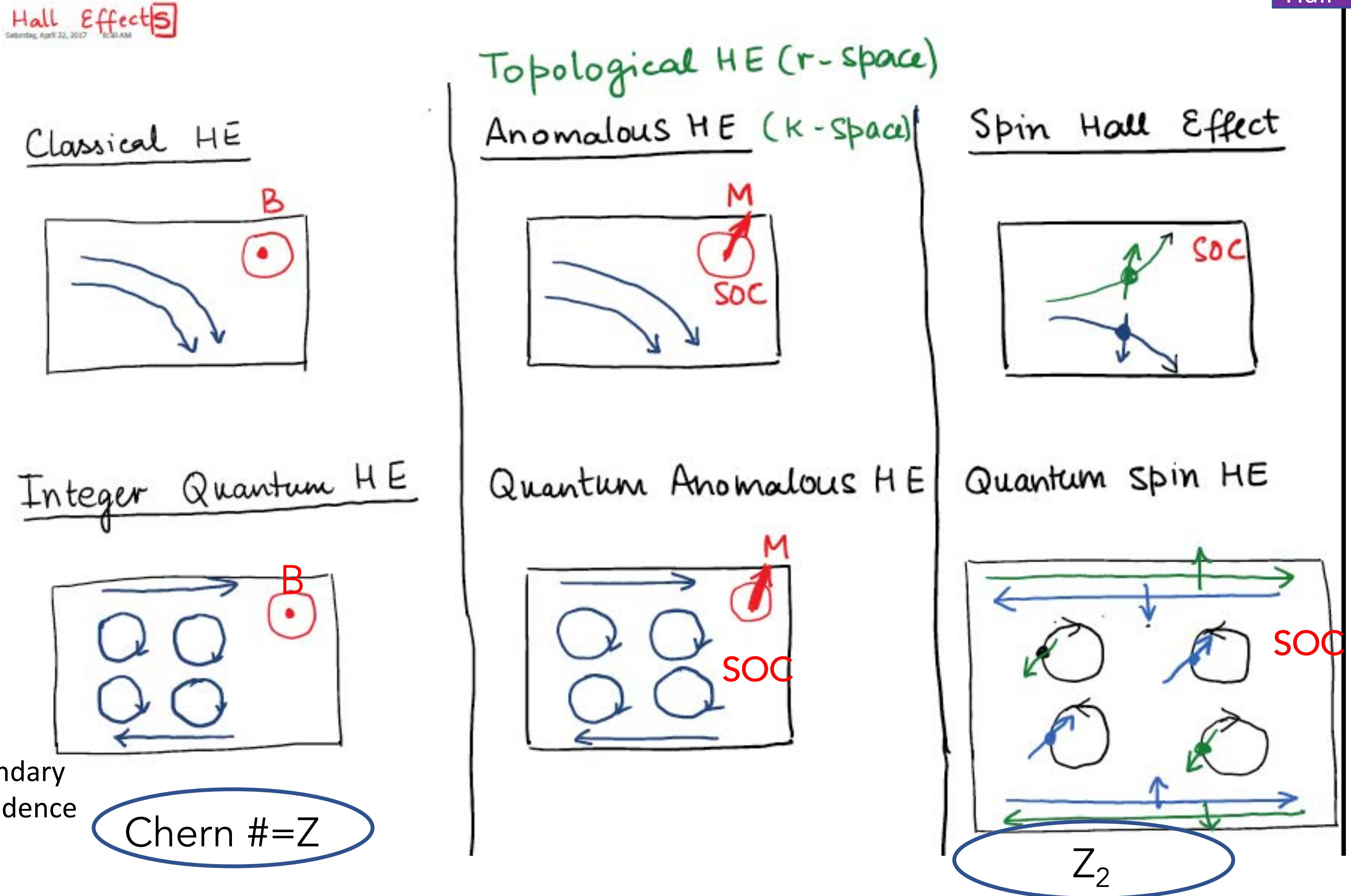
What is a topological insulator?

Bulk insulator
Conducting edge or surface states
Does not break TR



Bulk- Boundary correspondence

Topology and Topological Insulators

- Topology and Invariants: Gauss Bonnet formula
- Dynamic (usual) vs Geometric Phase
- Berry phase (topological invariant), can be measured
- Chern number and Quantum Hall Effect
- Degenerate bands, Berry Monopoles and Chern Number
- Spin Hall Effect

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Toys: Playdoh

$\text{cup} \simeq \text{donut}$

Topology and geometry: Gauss-Bonnet formula

- Topological (quantized) numbers can be written as integrals of local quantities

- Gauss-Bonnet theorem. The Gaussian curvature K of a 2D surface M of genus g integrated over the surface gives the Euler characteristic $\chi = 2 - 2g$

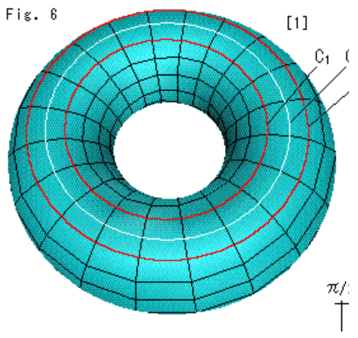
$$2 - 2g = \frac{1}{2\pi} \int_M K dA$$

- Can locally change the curvature, but the integral is quantized

- Relate such integrals to quantized Hall effects

$$K = \frac{1}{R^2}, \quad dA = R^2 d\Omega,$$

$$\chi = 2, \quad g = 0$$



$$\chi = 0, \quad g = 1$$

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Topology and band theory

- Bloch theorem: states are labeled by a crystal momentum ***k*** :

$$\psi_{n\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n\mathbf{k}}(\mathbf{r})$$

$u_{n\mathbf{k}}(\mathbf{r})$ • are lattice-periodic and are eigenstates of the Bloch Hamiltonian

$$H(\mathbf{k})|u_{n\mathbf{k}}\rangle = E_n(\mathbf{k})|u_{n\mathbf{k}}\rangle, \quad H(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{r}} H e^{i\mathbf{k}\cdot\mathbf{r}}$$

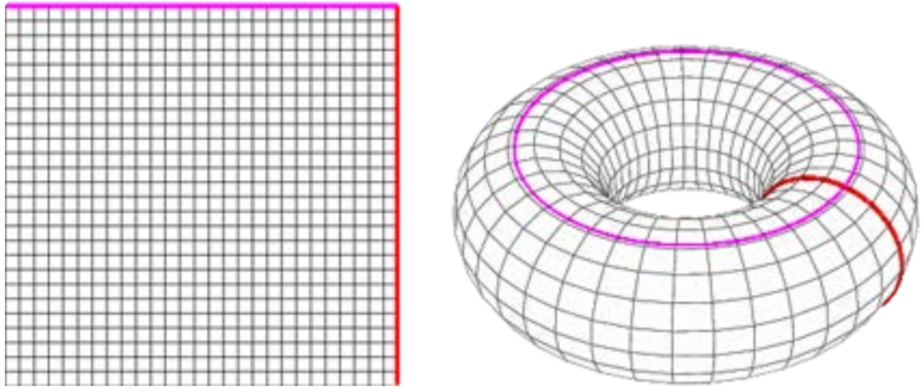
$\mathbf{k}$

n labels bands. Fully filled bands are separated by a gap from empty bands

- Lattice symmetry implies periodicity in the reciprocal (momentum) space

$$H(\mathbf{k}+\mathbf{G}) = H(\mathbf{k}) \quad \Rightarrow \quad \mathbf{k}+\mathbf{G} \equiv \mathbf{k}$$

- Crystal momenta lie in a periodic Brillouin zone



$\mathcal{BZ} \simeq \mathbb{T}^d$

gruzberg

Main question: Consider an electron spin in a magnetic field:

2 level system

Saturday, April 22, 2017 3:05 PM

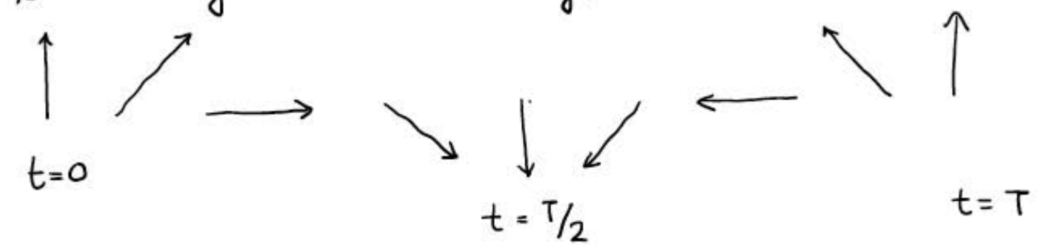
e in a magnetic field

$$\mathcal{H} = \vec{\sigma} \cdot \vec{B}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

PAULI MATRICES

Suppose the magnetic field keeps the same magnitude but changes its direction



After time $t = T$ the $\vec{B}(t)$ returns to its original value.

What is the net phase picked up by the electron $\gamma = 0$?

At each time t the electron spin aligns itself to the local "up" direction.

Berry phase

Saturday, April 29, 2017 10:01 AM

Phase acquired by wave function as the Hamiltonian is changed adiabatically

Remain on same eigenindex

$H(\alpha)$ ⟶ set of parameters $\{\alpha_1, \cdots \alpha_m\} \in M$

e.g $\{\vec{\alpha}\} \to \vec{k}$ in the BZ

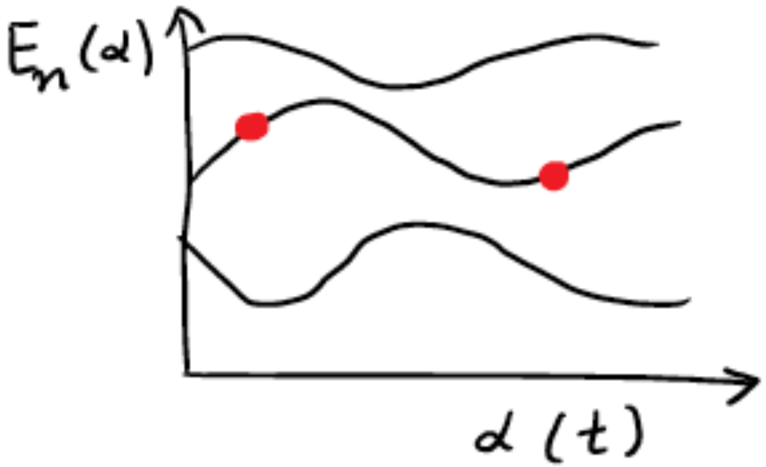
$M \simeq T^d$

$|n(\alpha)\rangle$ is the n^{th} eigenstate for a set of parameters α

$$\mathcal{H}(\alpha)|n(\alpha)\rangle = E_n(\alpha)|n(\alpha)\rangle$$

$|\Psi_n(k)\rangle$

Consider the evolution of the state $|n(\alpha)\rangle$ as the parameters $\alpha(t)$ evolve in time.



Adiabatic

time scale for evolution of $\alpha(t)$ is slow compared to energy gaps.

$|n(\alpha(t))\rangle$ remains an eigenstate but the wavefunction acquires a phase

$$|\psi_n(t)\rangle = e^{i\theta_n(t)} \, e^{i\gamma_n(t)} \, |n(\alpha(t))\rangle$$

Dynamic "usual" phase:

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(\alpha(t'))\, dt'$$

Geometric phase:

$$\gamma_n(t) = i\int_0^t \langle n(\alpha(t'))|\frac{d}{dt'}|n(\alpha(t'))\rangle dt'$$

(real)

* assume state is non-degenerate

For derivation: see NT notes
Also: Ballentine
"Quantum Mechanics"

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Berry connection

overlap of two wavefunctions infinitesimally separated in α - space

$$\langle n(\alpha) | n(\alpha+\Delta\alpha)\rangle = 1 + \Delta\alpha \langle n(\alpha)| \nabla_\alpha |n(\alpha)\rangle$$

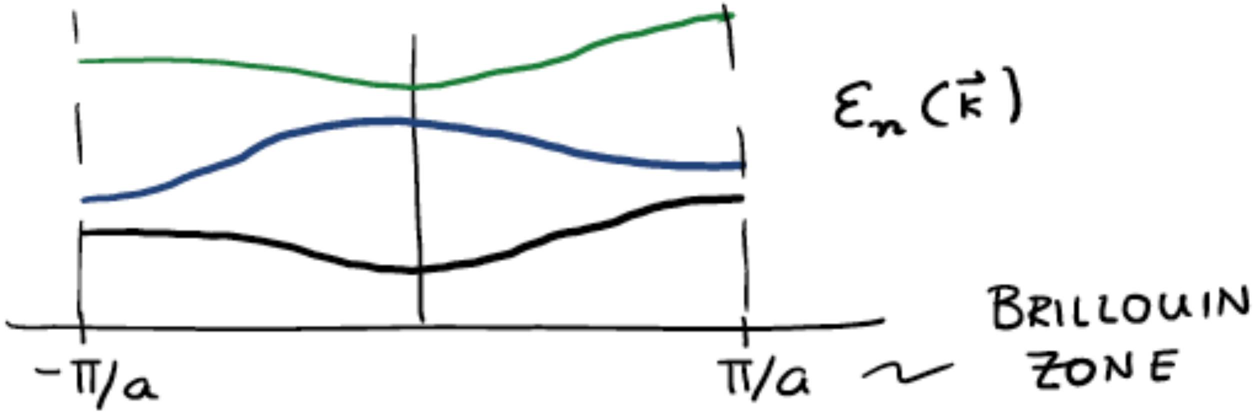
$$\simeq e^{-i\Delta\alpha \cdot \vec{a}_n(\alpha)}$$

vector potential

$$\vec{a}_n(\alpha) = i\langle n(\alpha)| \nabla_\alpha |n(\alpha)\rangle$$

Specific case: **Band structure of xtals**

$\vec{\alpha} = \vec{k}$ crystal momentum



Bloch wave function:

$$\psi_{n\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} \underbrace{u_{n\vec{k}}(\vec{r})}_{\text{periodic part}}$$

↑ band index

As $\vec{k}$ changes we map out an energy band

set of all bands → band structure

Brillouin zone will play the role of surface over which integrals will be calculated.

As $\vec{k}$ is changed over the Brillouin zone the phase picked up by the electron is

$$\gamma_n = \oint_{loop} \vec{a}_n \cdot d\vec{k} = \int_S \vec{\Omega}_n \cdot d\vec{S}$$

BERRY PHASE = total Berry flux within a Brillouin zone

|| geometric phase around loop

$$a_{n\mu}(\vec{k}) = i \langle u_{n\vec{k}} | \frac{\partial}{\partial k_\mu} | u_{n\vec{k}} \rangle$$

↑ along μ direction

$$\vec{a}_n(\vec{k}) = i \langle n\vec{k} | \vec{\nabla}_k | n\vec{k} \rangle$$

- Berry phase is gauge invariant and is measurable.

[First derived by Pancharatnam (1955) in optics]

Berry Flux

$$\vec{\Omega}_n = \vec{\nabla} \times \vec{a}_n$$

Chern # $C_n = \frac{\gamma_n}{2\pi}$

$$N = \sum_{n=}^{\# \text{ occ bands}} C_n$$

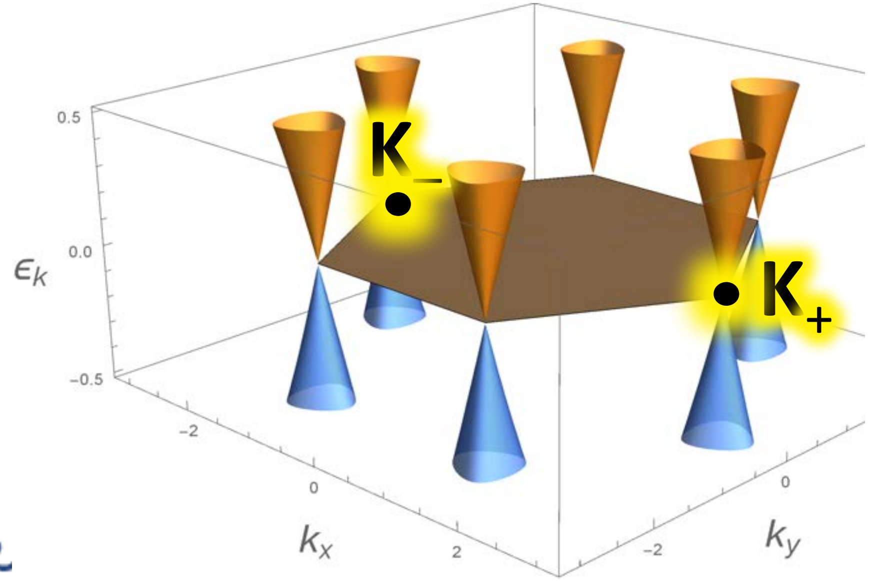
$$\sigma_{xy} = \left(\frac{e^2}{h}\right) N$$

Dictionary

Aharonov Bohm

Quantity	AB	Berry
vector potential	$\vec{A}(\vec{r})$	$\vec{A}_n(\vec{k}) = \langle u_{n\vec{k}} \vert i \vec{\nabla}_k u_{n\vec{k}} \rangle$ [Berry connection]
magnetic field	$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$	$\vec{\Omega}_n(\vec{k}) = \vec{\nabla}_k \times A_n(\vec{k})$ [Berry curvature]
flux	$\Phi = \oint_\ell \vec{A} \cdot \vec{d\ell} = \int_S \vec{B} \cdot \vec{da}$	$\Phi_n = \int_\ell \vec{A}_n(\vec{k}) \cdot \vec{dk} = \int_S \vec{\Omega}_n(\vec{k}) \cdot \vec{da}$ [Berry flux]
integrated phase	$\gamma = 2\pi \, \Phi / \phi_0$ $\phi_0 = h/e$	$\gamma_n = 2\pi \, \Phi_n / \phi_0$ [Berry phase] $C_n = \Phi_n / \phi_0$ [Chern #]

Degeneracies and two level systems



Berry curvature $\vec{\Omega}_n(\vec{k})$ is large near degeneracy points $\vec{k}_0$ where energy levels cross.

$$E_n(\vec{k}_0) = E_m(\vec{k}_0)$$

generically when two levels cross denote a

$$E_+(\vec{k}) \geq E_-(\vec{k})$$

Expand $\mathcal{H}(\vec{k}) \approx \mathcal{H}(\vec{k}_0) + (\vec{k}-\vec{k}_0)\cdot\vec{\nabla}\mathcal{H}(\vec{k})$

$$\vec{\Omega}_+(\vec{k}) = i\,\frac{\langle +,k|\vec{\nabla}H(k_0)|-,k\rangle \times \langle -,k|\vec{\nabla}H(k_0)|+,k\rangle}{\left(E_+(\vec{k}) - E_-(\vec{k})\right)^2}$$

$$\vec{\Omega}_+(\vec{k}) = -\vec{\Omega}_-(\vec{k})$$

- Berry curvature :

$$\vec{\Omega}_+(\vec{k}) = \frac{i \langle +,\vec{k} | \vec{\sigma} | -,\vec{k} \rangle \times \langle -,\vec{k} | \vec{\sigma} | +,\vec{k} \rangle}{4k^2}$$

Choose $\hat{z}$ along $\vec{k}$

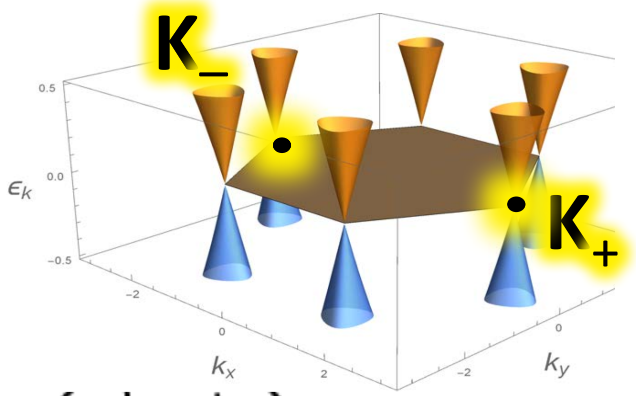
then

$$\sigma_z | \pm \rangle = \pm | \pm \rangle$$

$$\sigma_x | \pm \rangle = | \mp \rangle$$

$$\sigma_y | \pm \rangle = \pm i | \mp \rangle$$

$$\sigma_y |\pm\rangle = \pm i |\mp\rangle$$



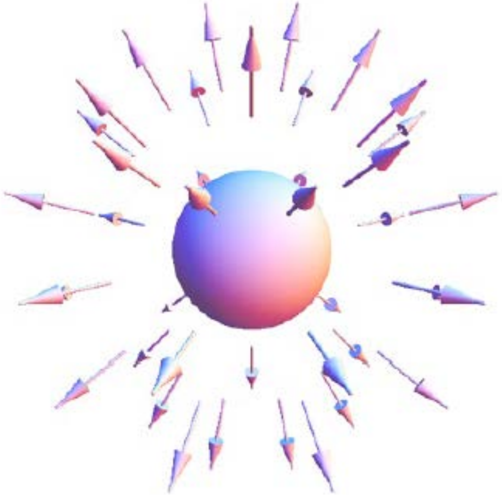
$$\Omega_{+,x} = \Omega_{+,y} = 0$$

(proportional to $\langle - | \sigma_z | + \rangle = 0$)

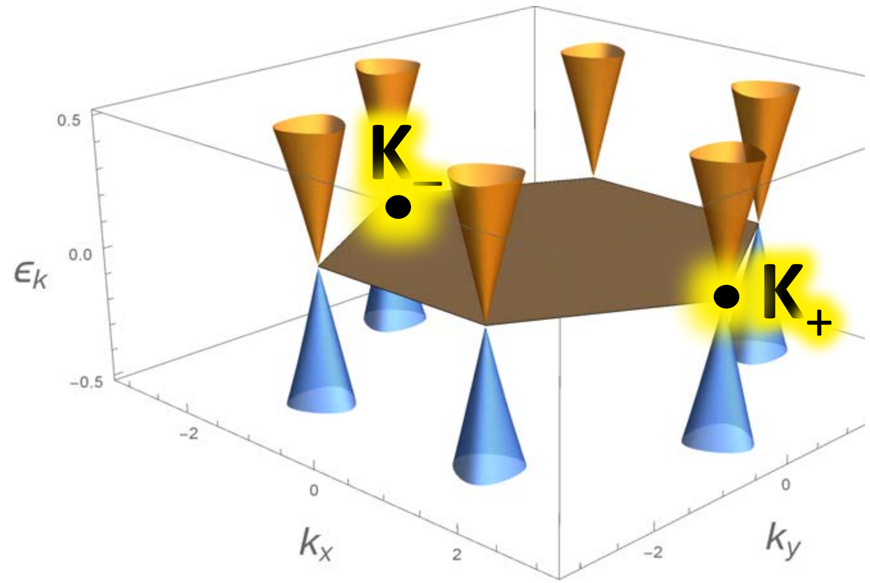
$$\Omega_{+,z} = i \frac{\langle + | \sigma_x | - \rangle \langle - | \sigma_y | + \rangle - \langle + | \sigma_y | - \rangle \langle - | \sigma_x | + \rangle}{4k^2}$$

$$\Omega_{+,z} = -\frac{1}{2k^2}$$

$$\vec{\Omega}_+ = -\frac{\vec{k}}{2k^3} \qquad \Omega_- = \frac{\vec{k}}{2k^3}$$



Monopole with charge $\pm\frac{1}{2}$ at degeneracy point.

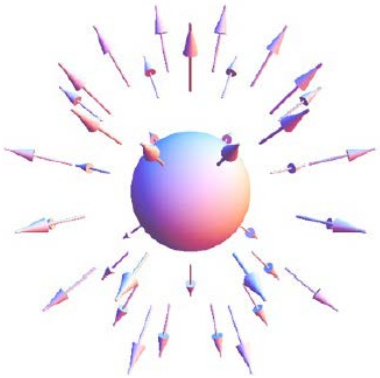


$$\vec{\Omega}_+ = -\frac{\vec{k}}{2k^3} \qquad \Omega_- = \frac{\vec{k}}{2k^3}$$

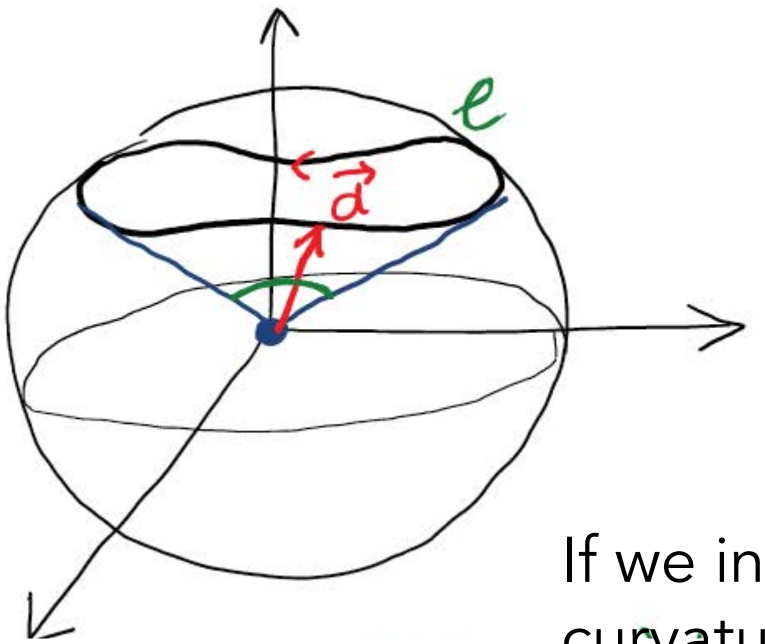
Monopole with charge $\pm \frac{1}{2}$ at degeneracy point.

Berry curvature is large near the degeneracy points

Berry curvature = Monopole



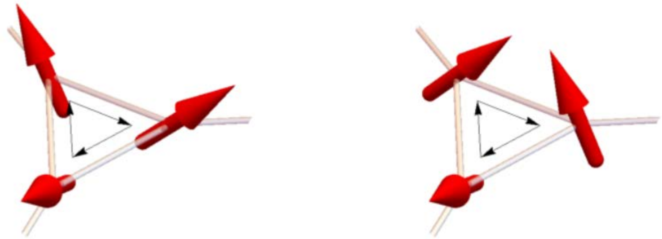
$$\gamma_{\pm}(\ell) = \int_S \vec{\Omega}_{\pm} \cdot d\vec{k} = \mp \frac{1}{2} \int_S \frac{\vec{k}}{k^2} \cdot d\vec{k} = \mp \frac{1}{2} \Big(\text{solid angle subtended}$$



If we integrate the Berry curvature over a sphere containing the monopole we get 2π

Chern # $\quad C_n = \dfrac{\gamma_n}{2\pi}$

d vector to Berry Curvature

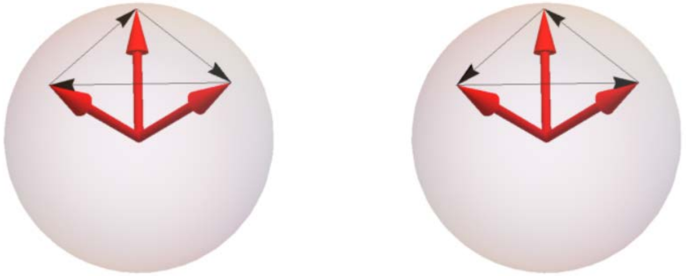


For a generic 2-level system
Berry flux (or Berry curvature)

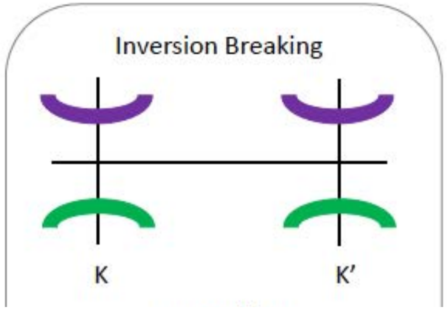
$$\vec{\Omega}_{\pm, jk} = \mp \frac{1}{2} \hat{d} \cdot (\partial_j \hat{d} \times \partial_k \hat{d})$$

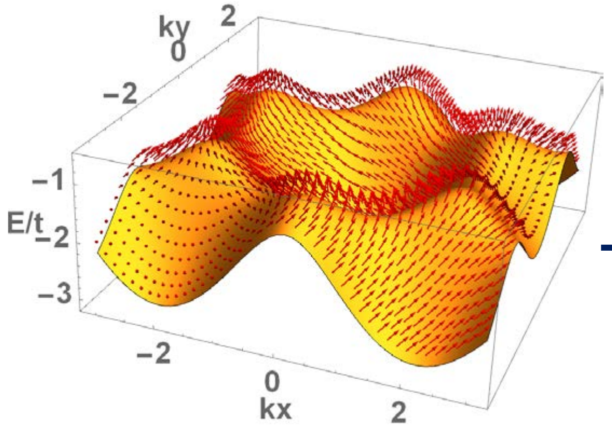
= solid angle on unit sphere $\hat{d}$

$\hat{d}(\vec{k})$ maps the manifold $M \rightarrow S^2$
e.g T^2
(torus)

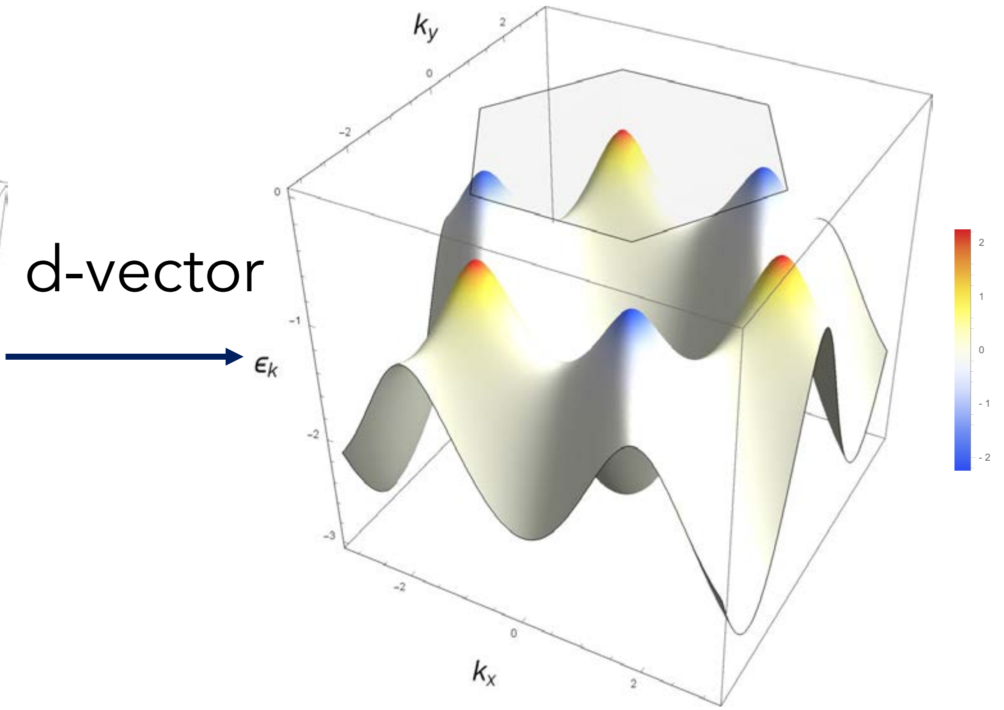


Honey comb with sublattice potential





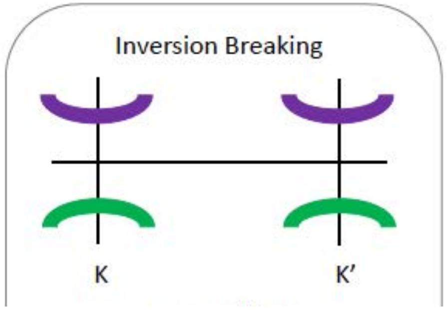
d-vector

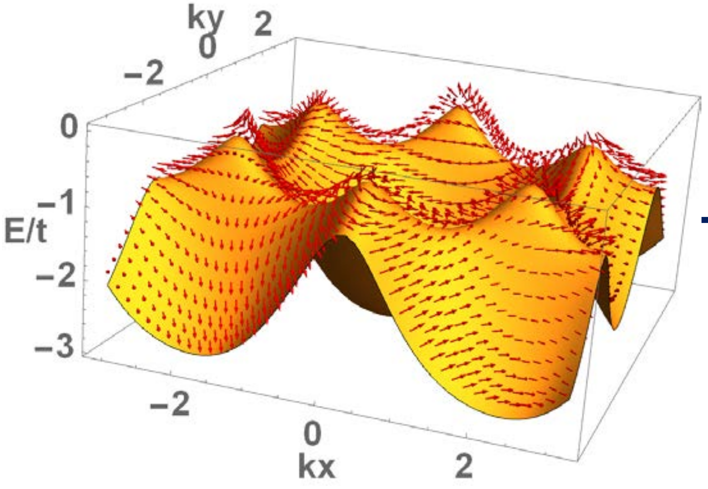


Berry curvature (analog of local magnetic field) on the lower band shows high density at the originally degenerate points. The mass terms have opened a gap at these points and created a monopole. However you see that the Berry curvature at the K and K' points have opposite sign so if you add up the total Berry curvature or total magnetic field on this band adds up to zero.

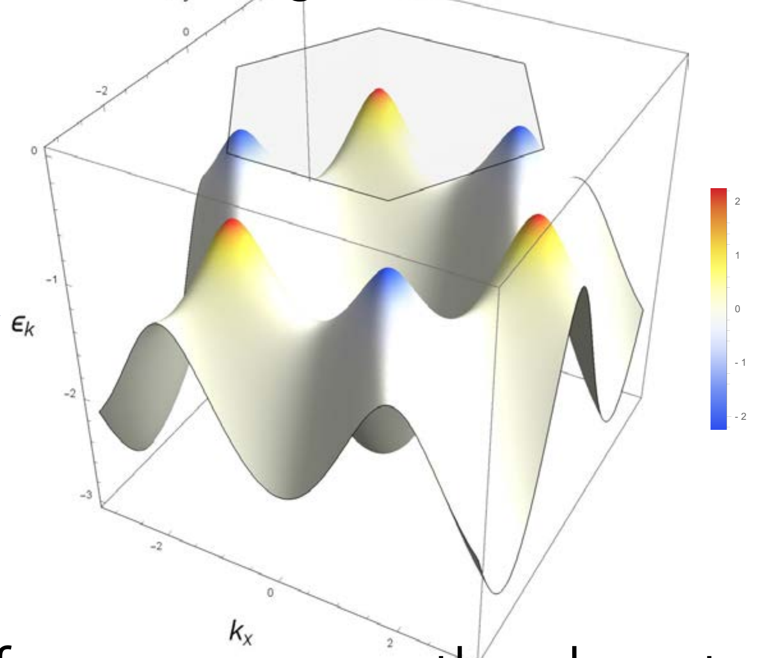
Honey comb with sublattice potential

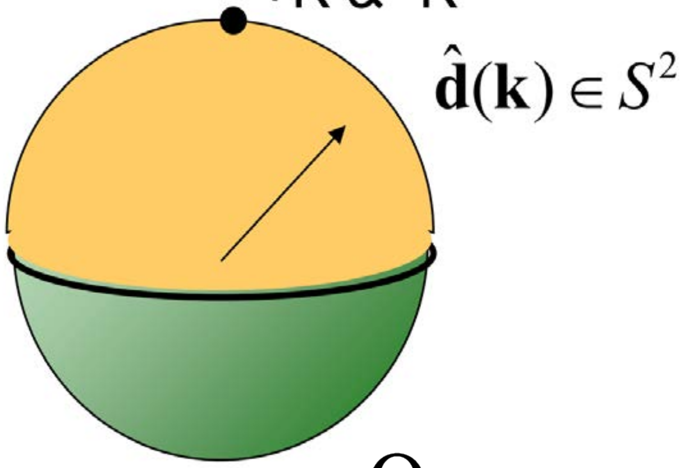
d-vector





Berry curvature
"magnetic field"

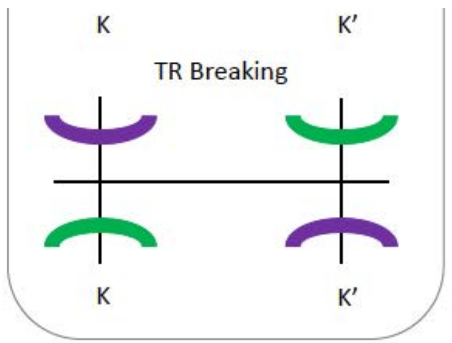




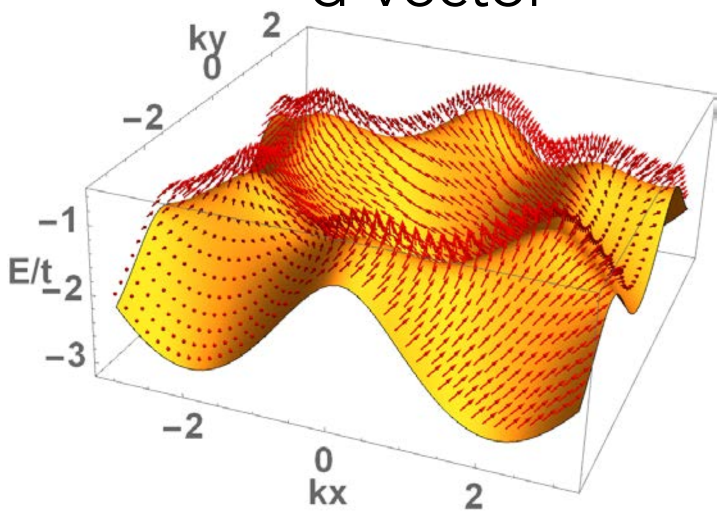
$$c_- = 0$$

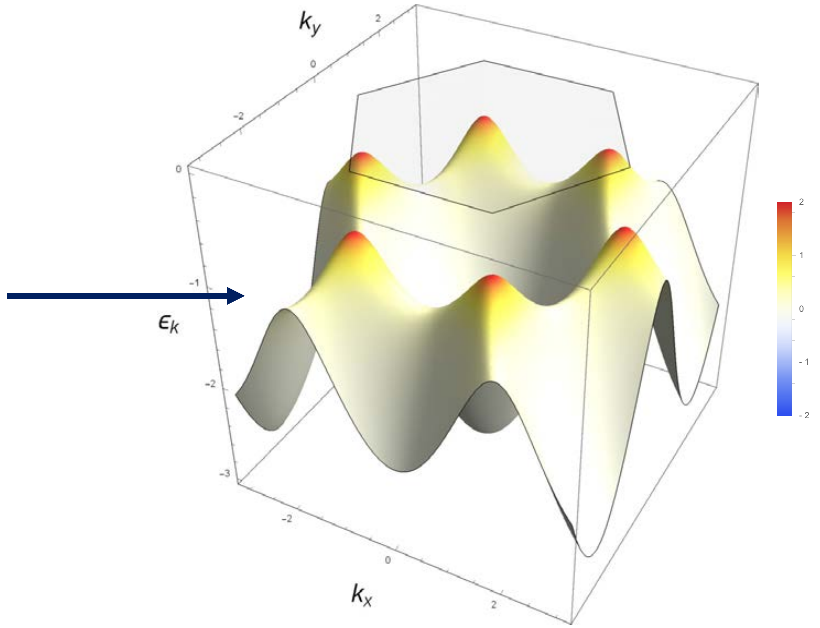
If you now map the d-vectors on the sphere they are essentially fluctuating around the north pole so the map can be continuously deformed to a point– the north pole– essentially the d-vectors can be combed straight up.

Honey comb with time reversal breaking

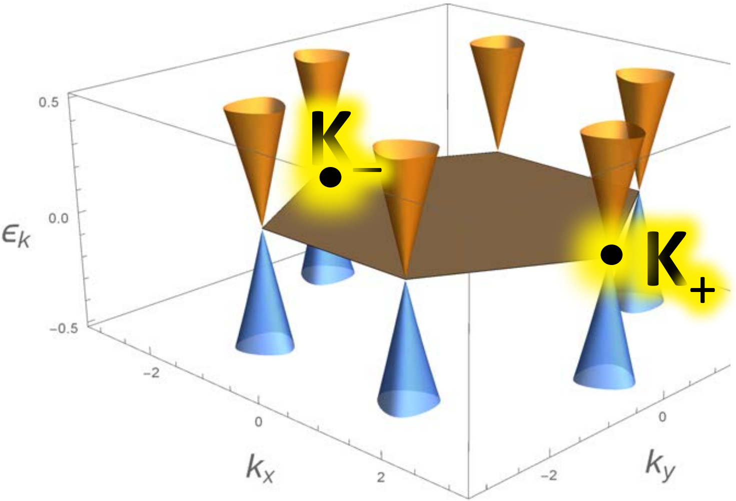


d-vector

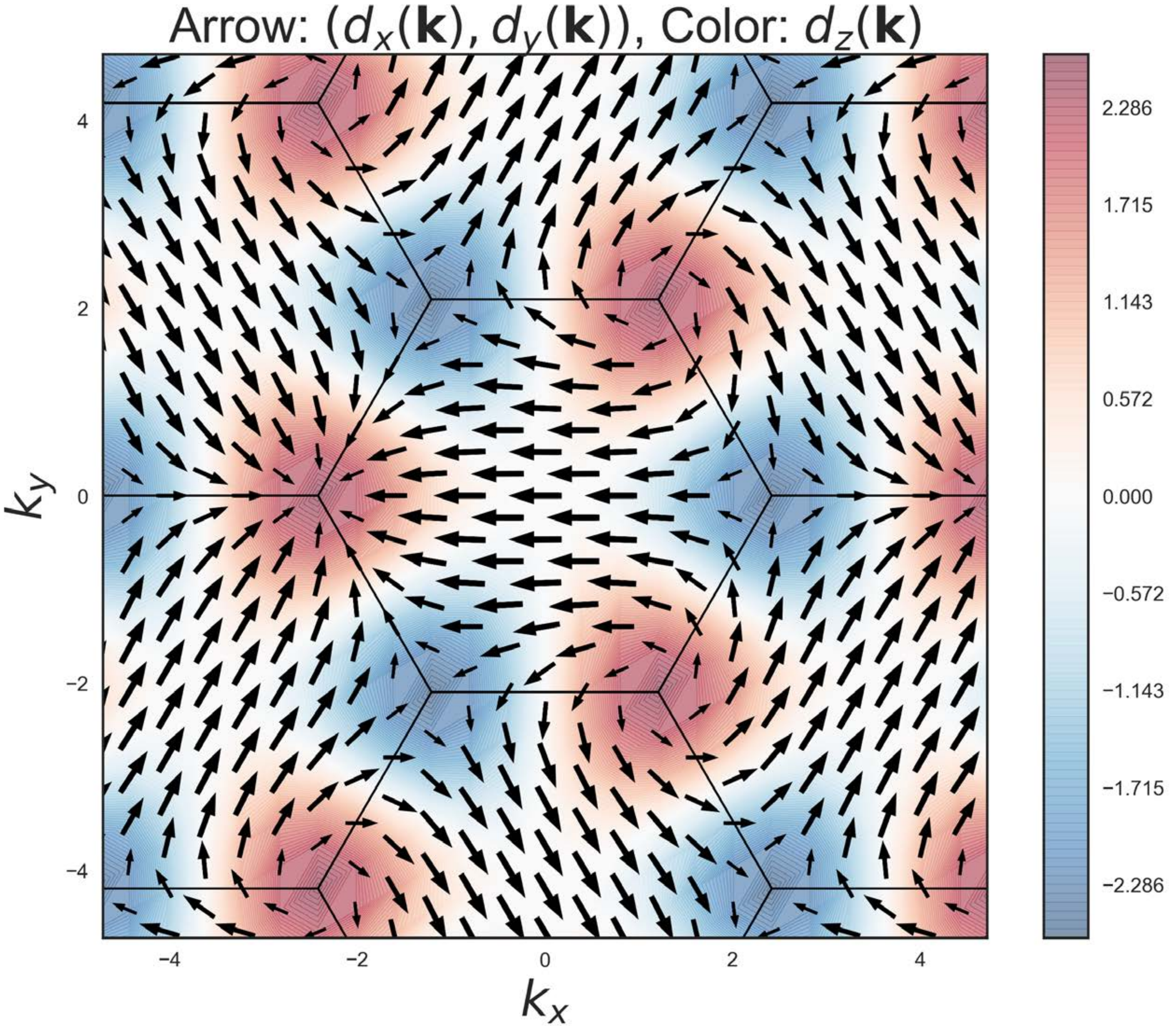




Berry curvature on the lower band shows high density at the originally degenerate points. The mass terms have opened a gap at these points and created a monopole. For the TR breaking case the Berry curvature at the K and K' points have the same sign so if you add up the total Berry curvature or total magnetic field on this band adds up to 1.

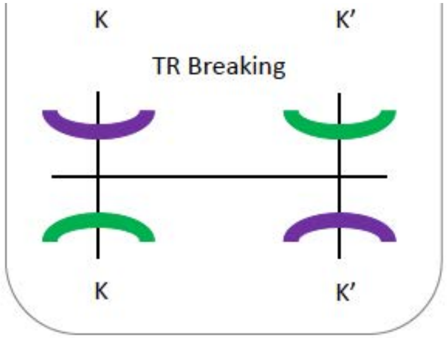


0.5
0.0
-0.5
ϵ_k
K_-
K_+
-2
0
2
k_x
-2
0
k_y

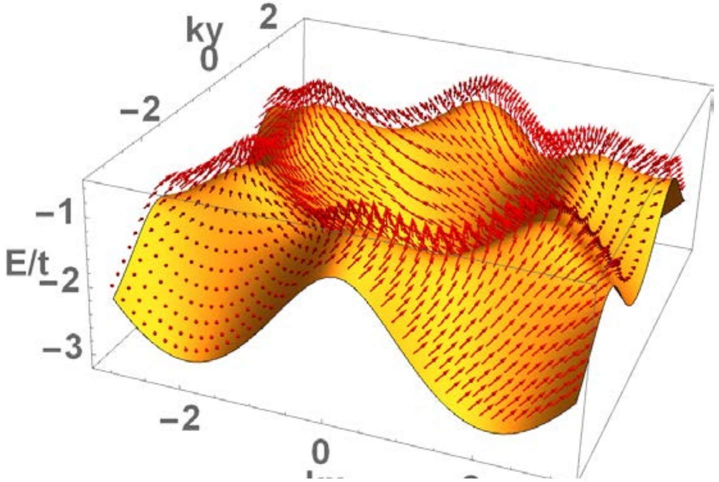


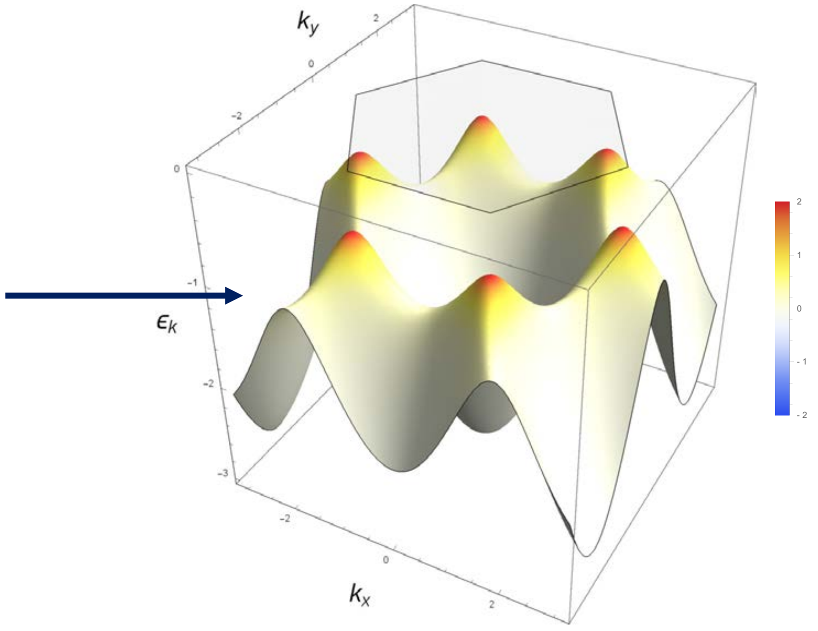
Arrow: $(d_x(\mathbf{k}), d_y(\mathbf{k}))$, Color: $d_z(\mathbf{k})$
k_y
4
2
0
−2
−4
k_x
−4
−2
0
2
4
2.286
1.715
1.143
0.572
0.000
−0.572
−1.143
−1.715
−2.286

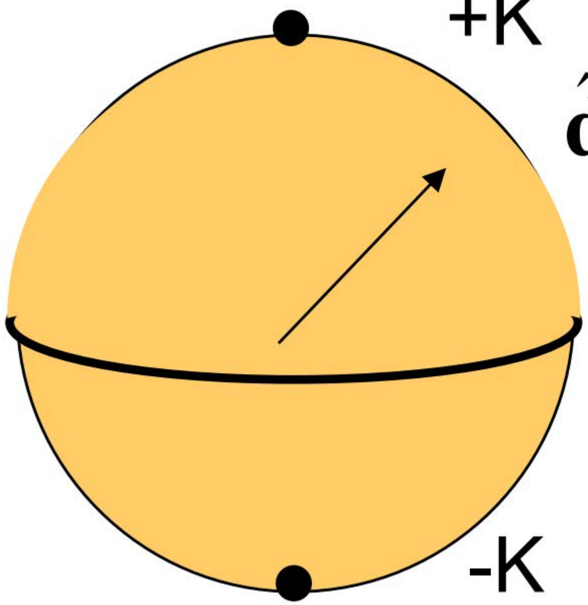
Honey comb with time reversal breaking



d-vector







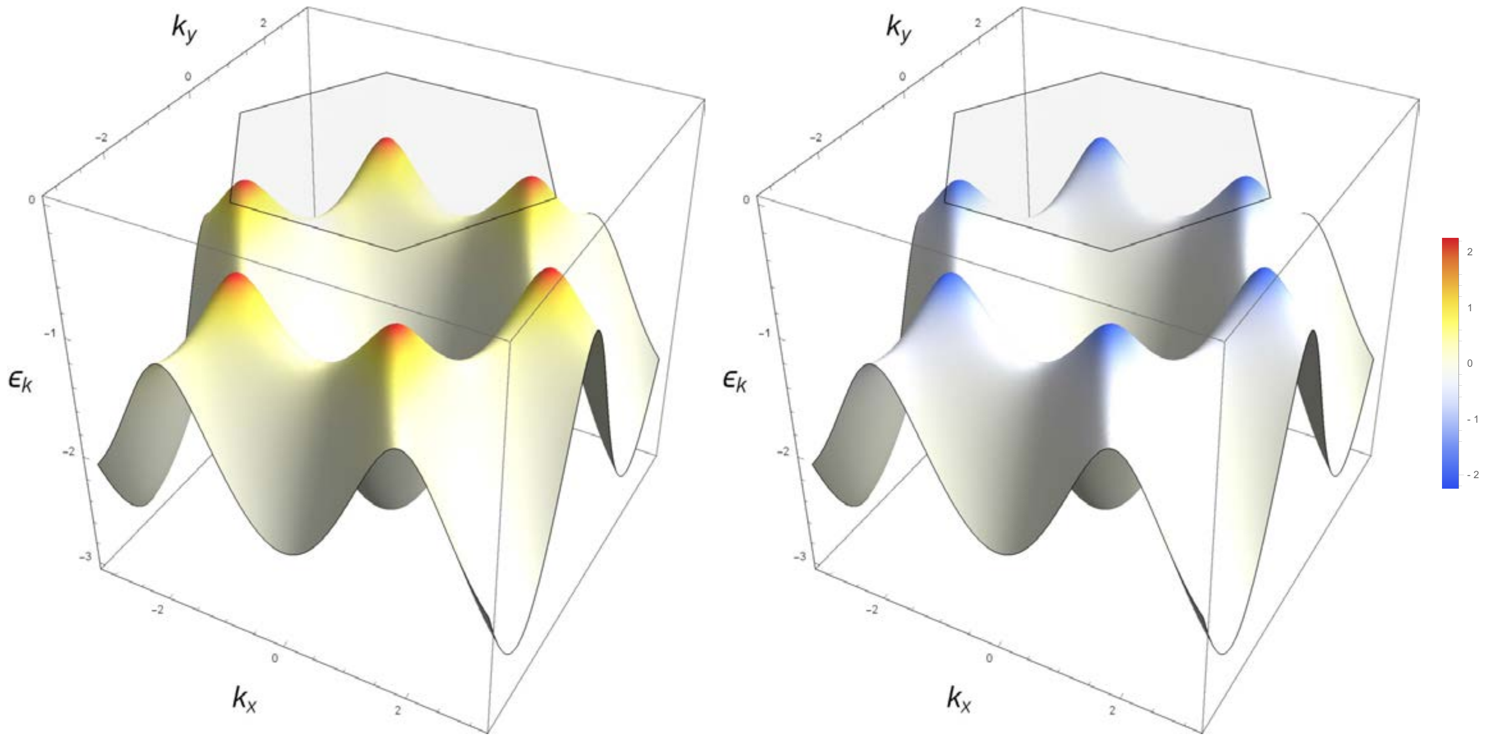
$\hat{\mathbf{d}}(\mathbf{k}) \in S^2$

$$c_- = 1$$

The d vectors around K map to the north pole, the d vectors around K' map to the south pole, The entire sphere is covered and that gives a Chern number of 1

Now include spin → spin hall effect:

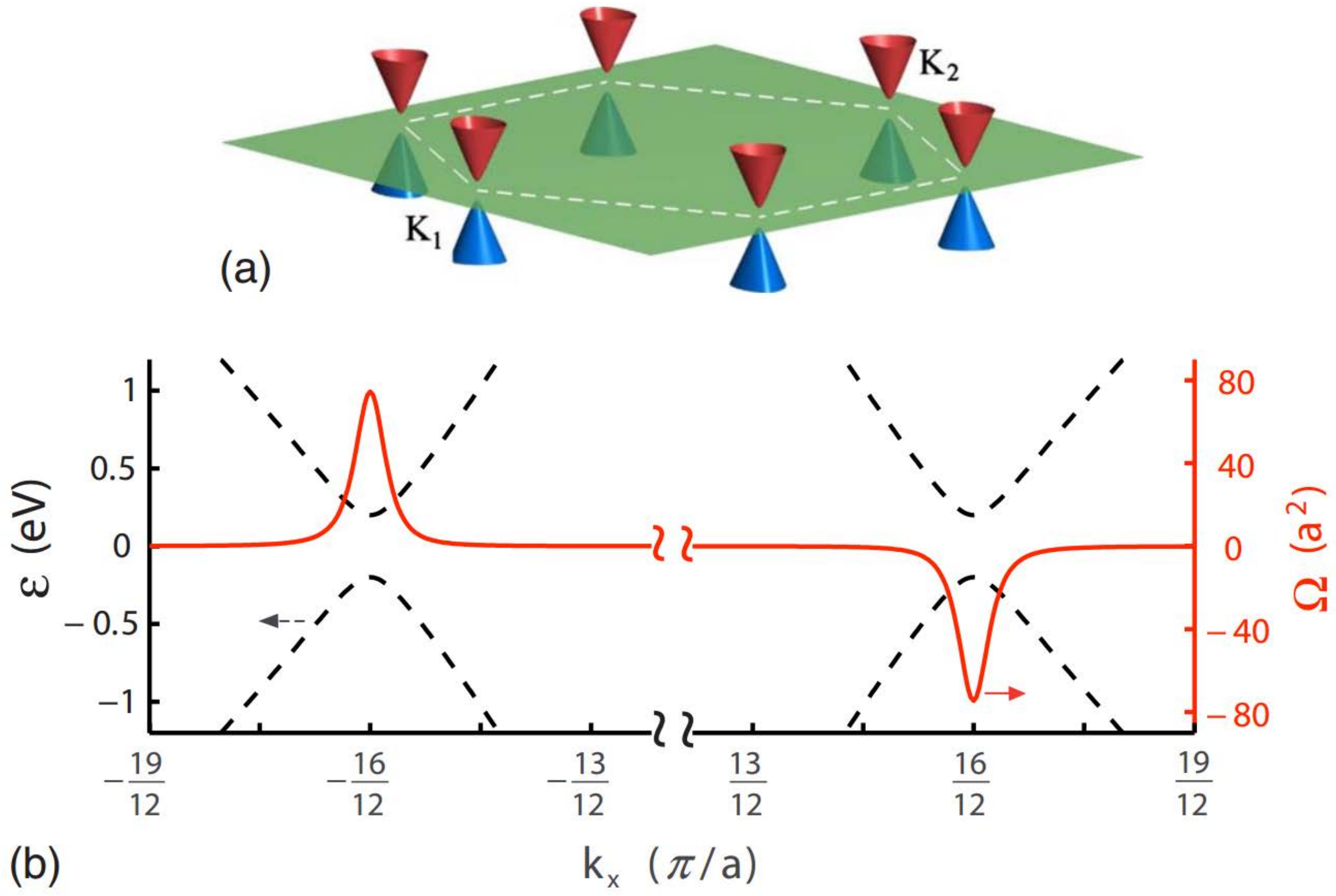
spin ↑ spin ↓

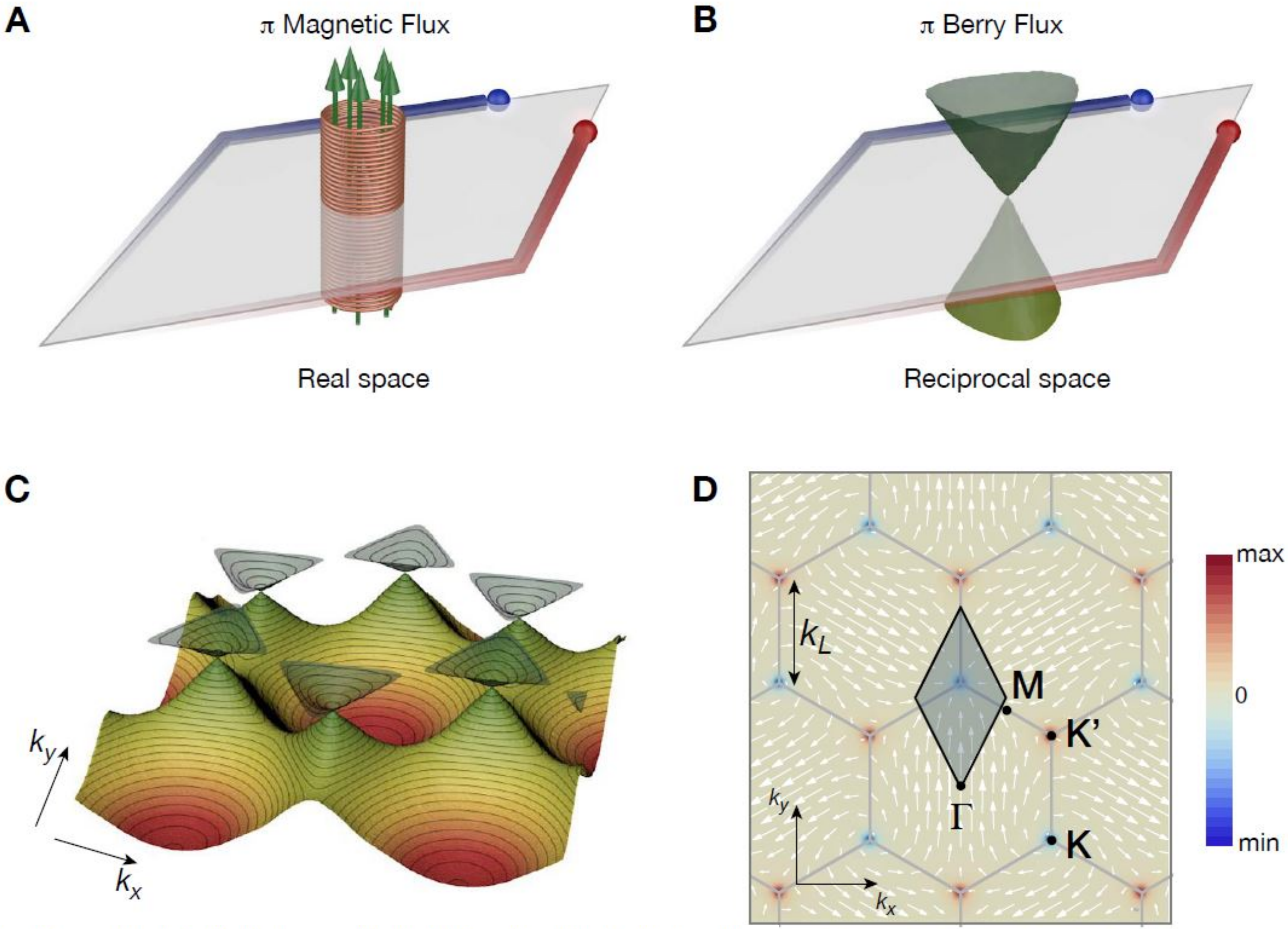


?

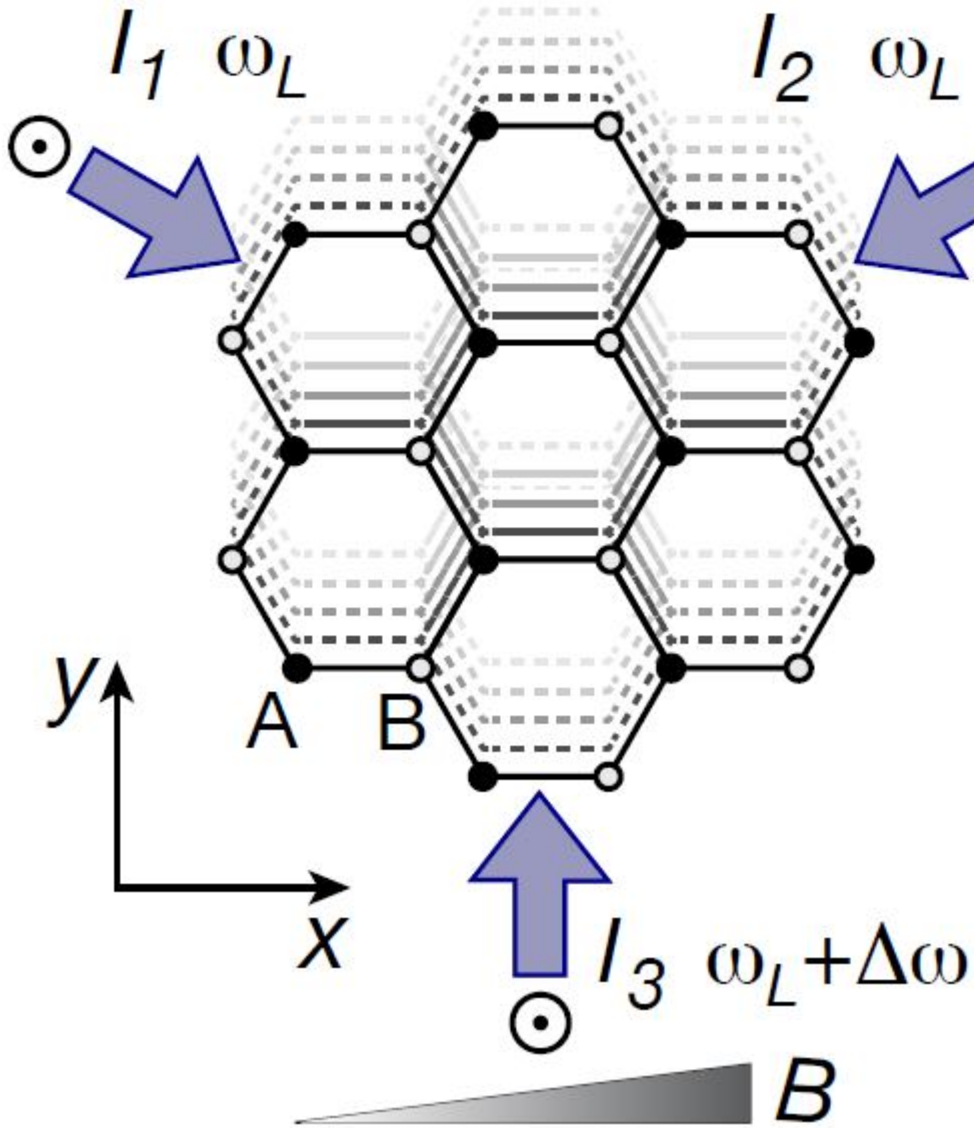
Can the Berry phase be measured? Directly?

Berry Curvature in Graphene





L. Duca, T. Li, R. Reitter, M. Schleier-Smith,U. Schneider,
Science **347**, 6219 (2015).



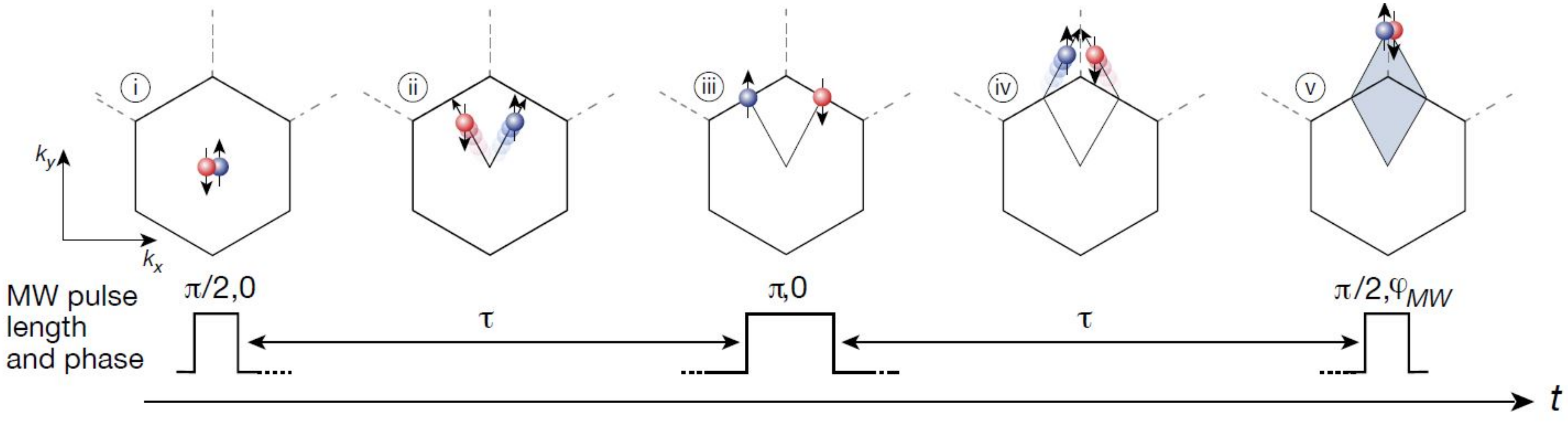
Ultracold ^{87}Rb in graphene-like honeycomb lattice

Three linearly polarized blue-detuned running waves at 120º at ω_L

Lattice acceleration from $\Delta\omega$ provides spin-independent force

B-field gradient creates spin-dependent force

L. Duca, T. Li, R. Reitter, M. Schleier-Smith, U. Schneider, *Science* **347**, 6219 (2015).



(i) Resonant $\pi/2$-pulse creates coherent superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$ states

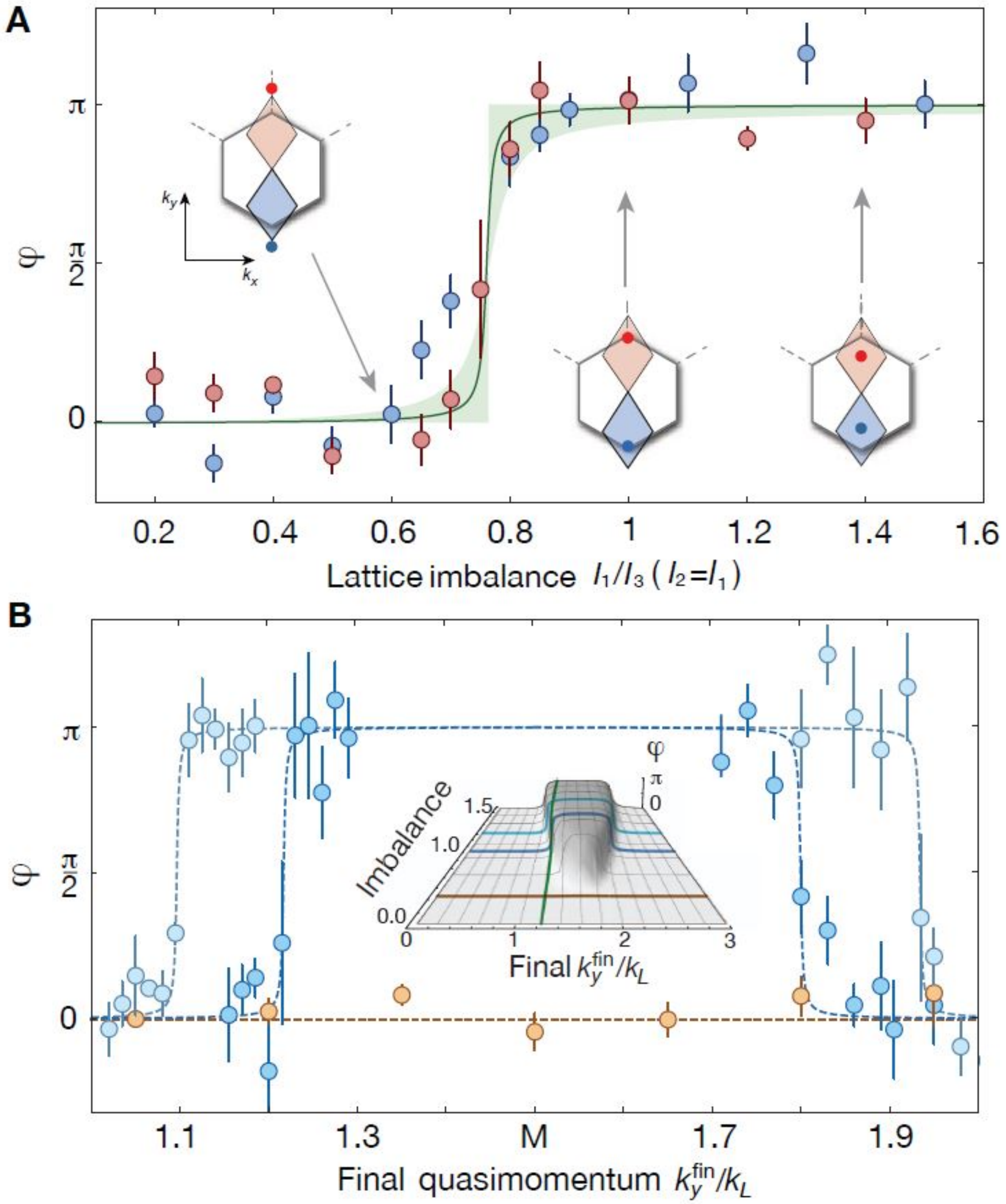
(ii) B-field gradient and lattice acceleration move atoms adiabatically along spin-dependent paths

(iii) π-pulse swaps the $|\uparrow\rangle$ and $|\downarrow\rangle$ states

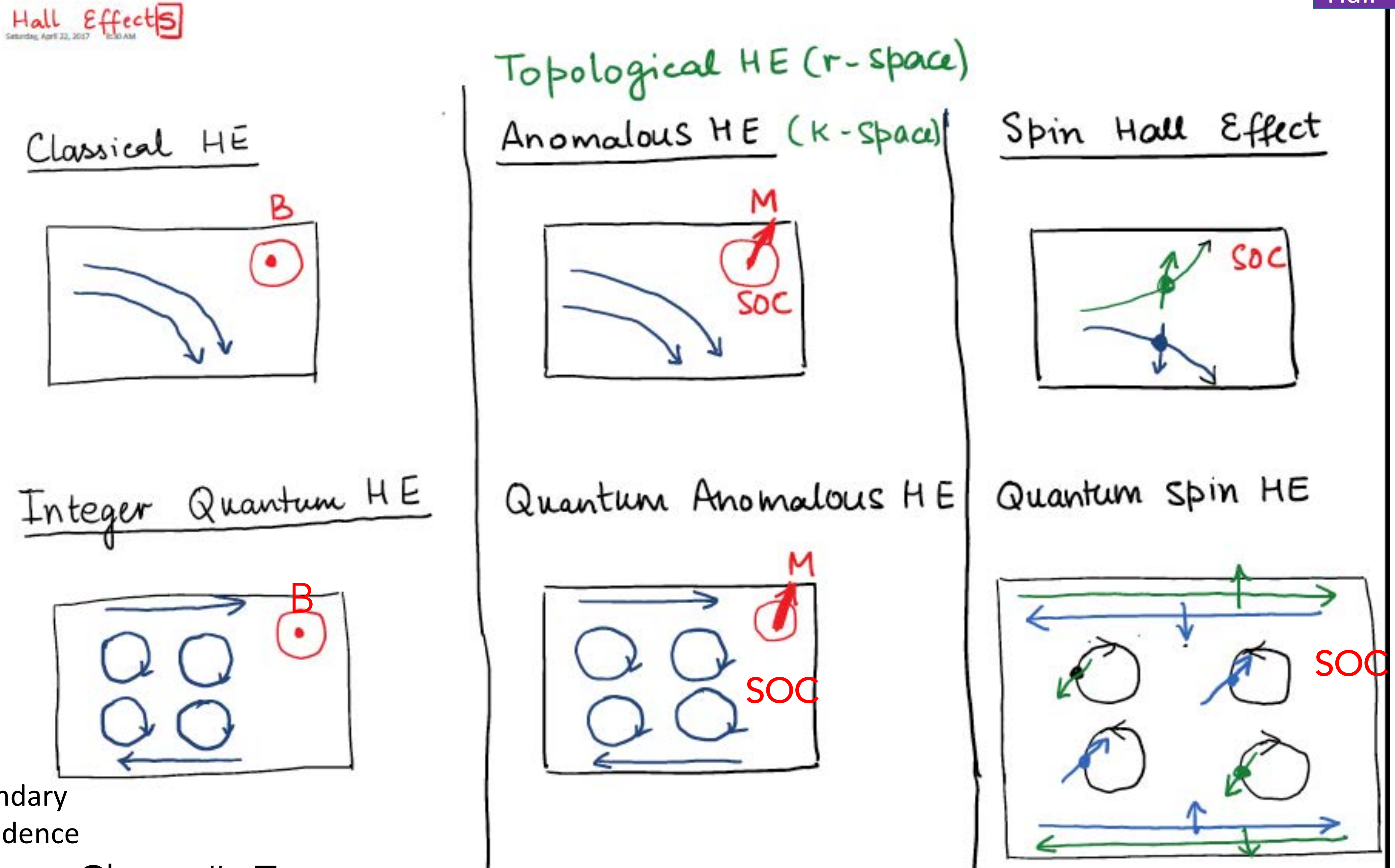
(iv) Each cloud experiences opposite force from B-field gradient in the x-direction

(v) Second $\pi/2$-pulse closes interferometer and converts phase information into spin populations $n_{\uparrow,\downarrow} \propto \cos(\phi + \phi_{MW})$

L. Duca, T. Li, R. Reitter, M. Schleier-Smith,U. Schneider, *Science* **347**, 6219 (2015).



L. Duca, T. Li, R. Reitter, M.
Science **347**, 6219 (2015).



Bulk- Boundary correspondence

Chern #=Z

Z_2

QH vs. QSH

Quantum Hall

Breaks TR Symmetry

Charge Conductivity Quantized:
$\sigma_H^C = \sigma_H^\uparrow + \sigma_H^\downarrow$

Magnetic Field

Topological Invariant:
Chern Number $\in \mathbb{Z}$

Quantum Spin Hall

TR Symmetry Required

Spin Conductivity Quantized:
$\sigma_H^S = \sigma_H^\uparrow - \sigma_H^\downarrow$

Spin-Orbit Coupling

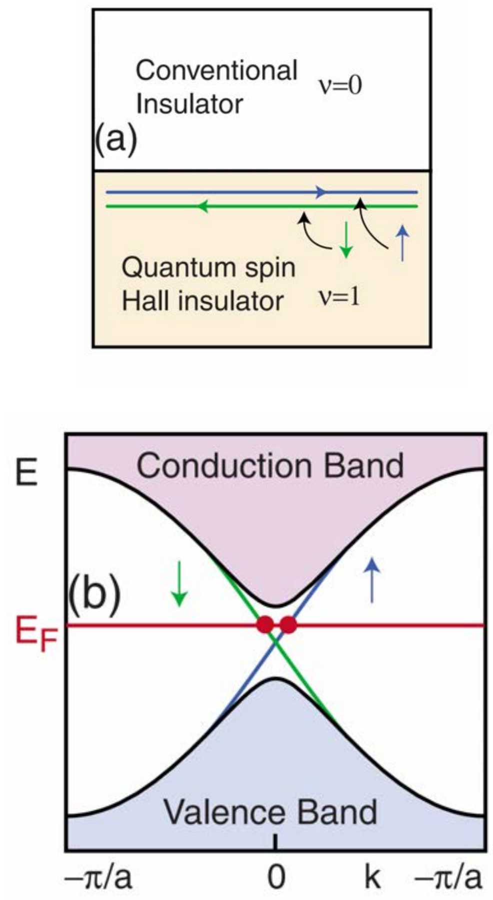
Topological Invariant:
Even/Odd $\in \mathbb{Z}_2$

Topological Insulator

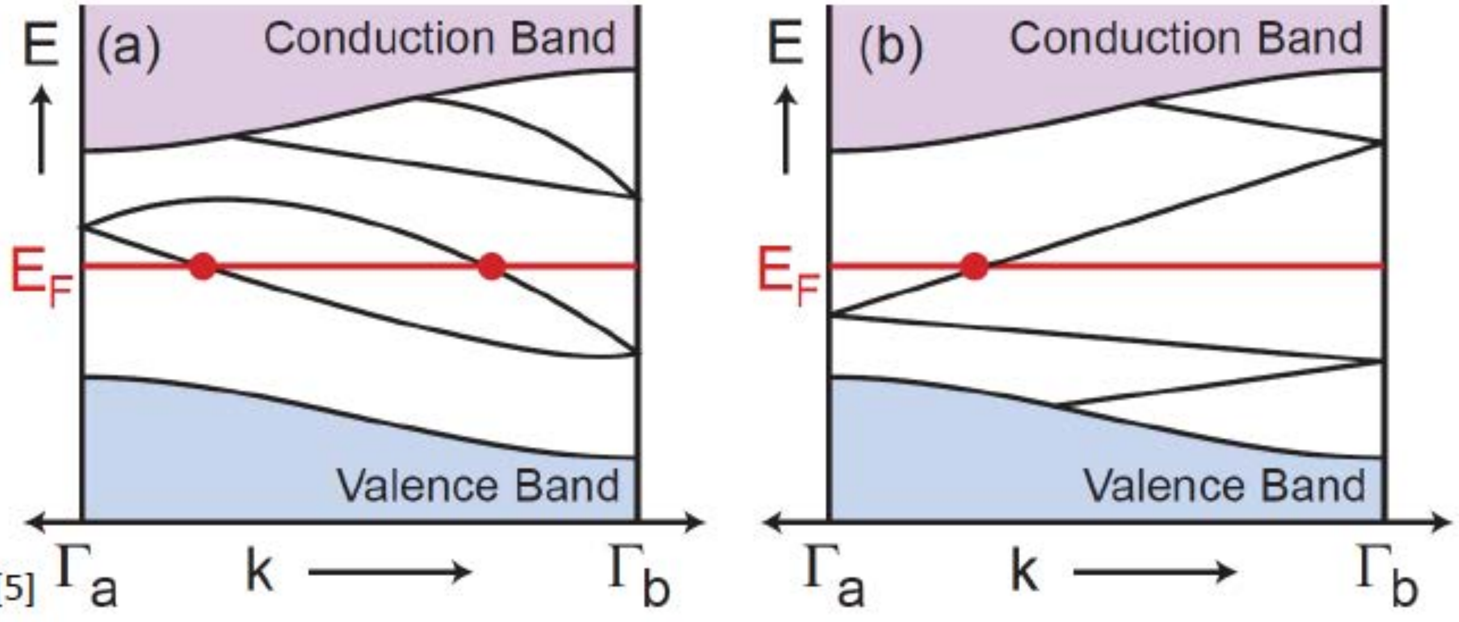
- if two inequivalent insulators are in contact with each other, the gap must vanish at the boundary.

- Gapless states must exist at the boundary between inequivalent insulators

- The gapless states can also be classified topologically using the bulk-boundary correspondence
- Topological Invariant: Z_2 index (odd/even)



Trivial Insulator: Z_2=even   Topological Insulator: Z_2=odd



[5]

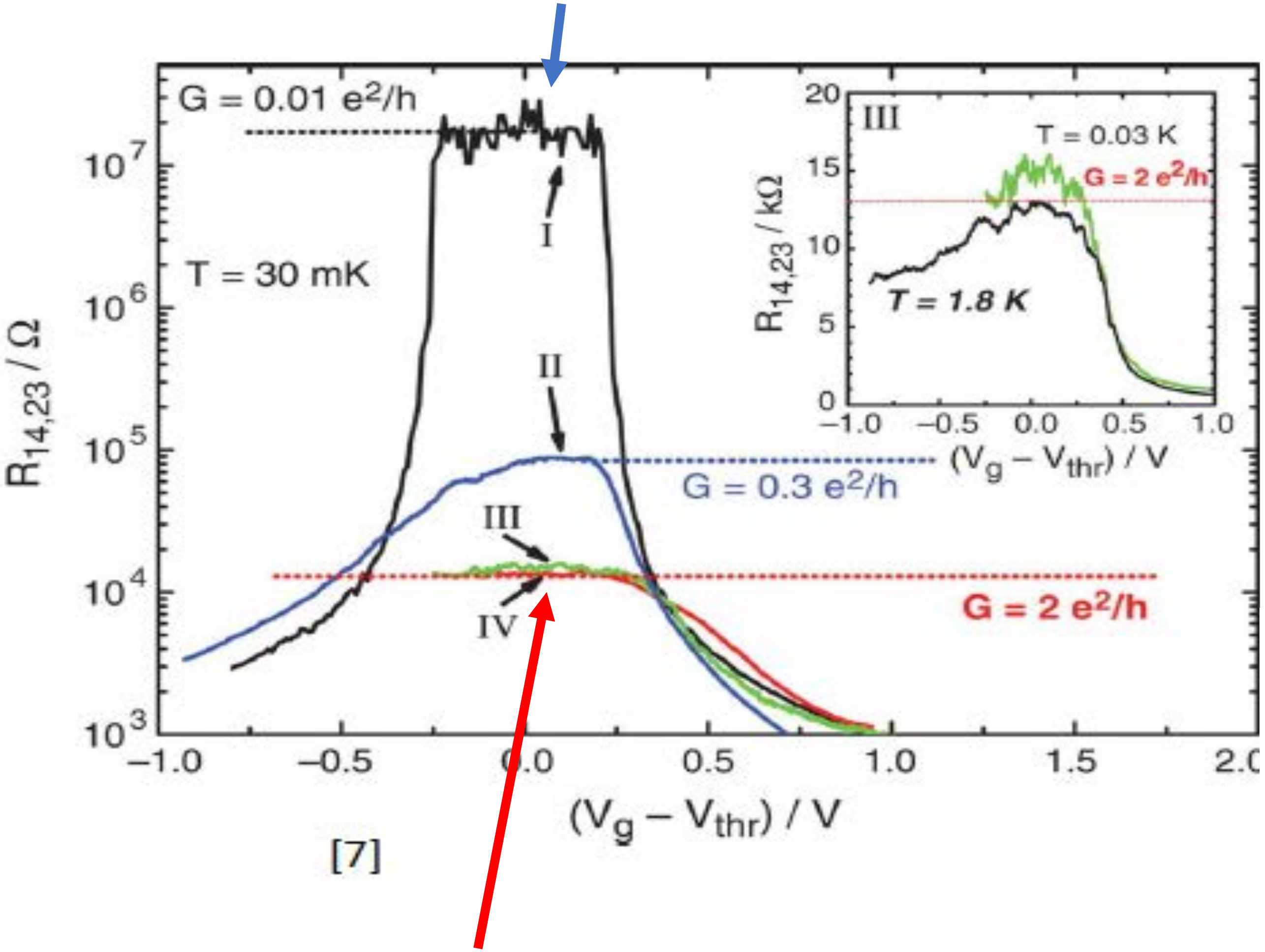
Kramer's Theorem → Degenerate Pairs at TR invariant momenta

Topological invariant is Z_2:
Either 0 or 1
Calculated as product of parity eigvalues at TRIM
Time reversal invariant momenta
(for systems with P symmetry)

On boundary the Z_2 index corresponds to the numbers of pairs of edge modes

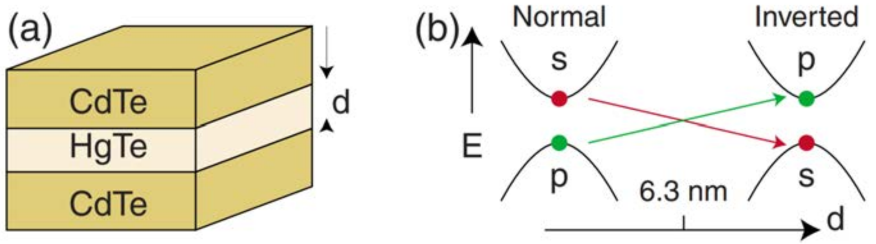
Quantum Spin Hall Effect:

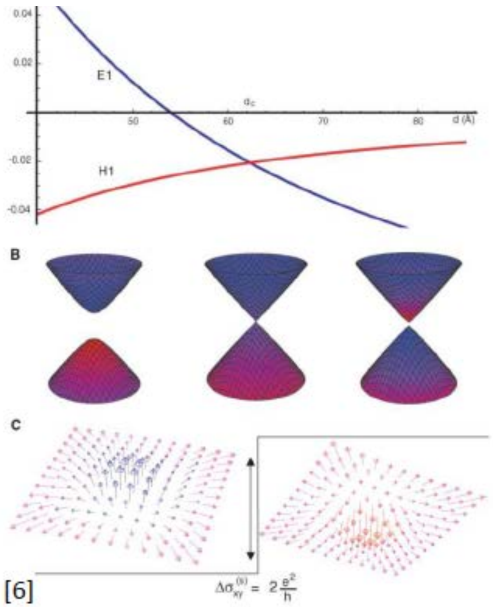
Trivial Insulator



[7]

Topological Insulator





[6]

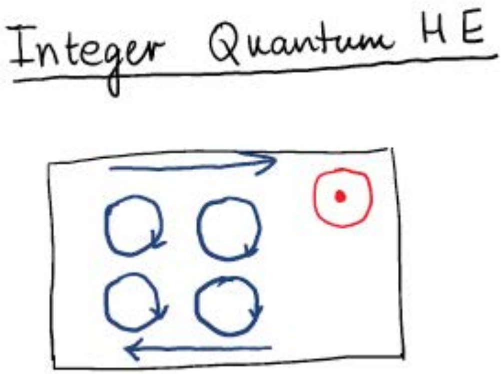
Band crossing as function of well width

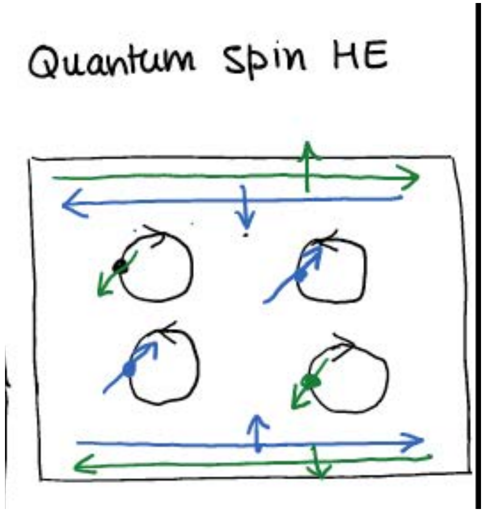
Crossing induces change of band character

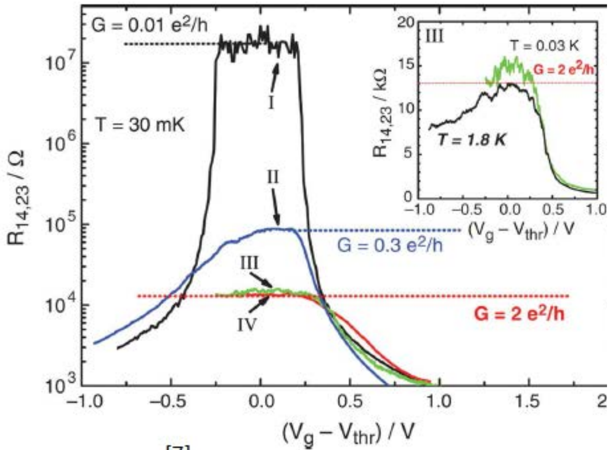
The parameter $\vec{d}(k)$

….*sheer poetry*

Idea→ prediction→ discovery → precise quantization







[7]

Topology and Topological Insulators

- Topology and Invariants: Gauss Bonnet formula
- Topology and Band Theory
- Dynamic (usual) vs Geometric Phase
- Berry phase (topological invariant), can be measured
- Chern number and Quantum Hall Effect
- Degenerate bands, Berry Monopoles and Chern Number
- Spin Hall Effect

2 level system

Saturday, April 22, 2017 3:05 PM

e in a magnetic field

$\mathcal{H} = \vec{\sigma} \cdot \vec{B}$

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

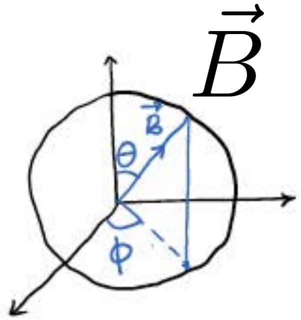
PAULI MATRICES

$$\vec{\sigma} \cdot \vec{B} = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix} \equiv \begin{pmatrix} B_z & B_- \\ B_+ & -B_z \end{pmatrix}$$

Eigenspectrum:

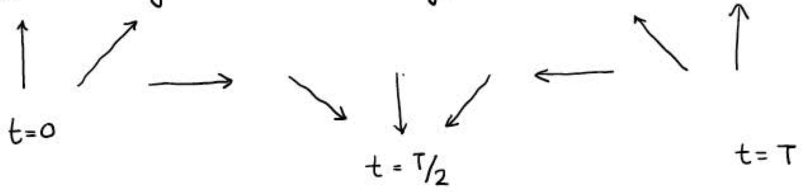
$\lambda_1 = +B \qquad \chi_+ = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \; e^{i\phi} \end{pmatrix}$

$\lambda_2 = -B \qquad \chi_- = \begin{pmatrix} \sin\theta/2 \; e^{-i\phi} \\ -\cos\theta/2 \end{pmatrix}$



Spinors

Suppose the magnetic field keeps the same magnitude but changes its direction



At each time t the electron spin aligns itself to the local "up" direction.

$$\theta = 0 \qquad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\theta = \pi/2, \phi = 0 \qquad \chi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Berry

After time $t = T$ the $\vec{B}(t)$ returns to its original value.

What is the net phase picked up by the electron

$\gamma = 0$?

Overlap of wave function at two nearby times:

$$\langle \chi_+(\theta, \phi) \,|\, \chi_+(\theta + \Delta\theta, \phi + \Delta\phi) \rangle$$

$$=$$

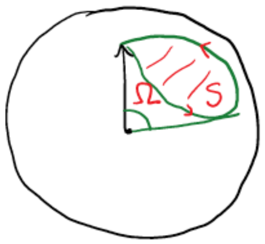
$$a_\phi = \langle \chi_+ | \vec{\nabla} \chi_+ \rangle = i \frac{\sin^2 \theta/2}{B \sin\theta} \hat{\phi}$$

$$\vec{\Omega} \equiv \vec{\nabla} \times \vec{a} = \frac{i}{2 r^2} \hat{r}$$

$$\gamma_+(T) = i \int_S \vec{\Omega} \cdot \underbrace{d\vec{S}}_{r^2 d\omega \hat{r}}$$

$$= -\frac{\Omega}{2}$$

Ω = solid angle subtended by the changing magnetic field over 1 time period.



Berry

To evaluate the Berry phase construct:

$$\nabla \chi_+ = \frac{\partial \chi_+}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \chi_+}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \chi_+}{\partial \phi}\hat{\phi}$$

$$\nabla \chi_+ = \frac{1}{r}\begin{pmatrix} -\frac{1}{2}\sin\theta/2 \\ \frac{1}{2}e^{i\phi}\cos\theta/2 \end{pmatrix}\hat{\theta} + \frac{1}{r\sin\theta}\begin{pmatrix} 0 \\ i\, e^{i\phi}\sin\theta/2 \end{pmatrix}\hat{\phi}$$

$$\langle \chi_+ | \nabla \chi_+ \rangle = \frac{1}{2r}\Big[-\sin\theta/2\,\cos\theta/2\,\hat{\theta} + \sin\theta/2\,\cos\theta/2\,\hat{\theta} + 2i\,\frac{\sin^2\theta/2}{\sin\theta}\hat{\phi}\Big]$$

$$\langle \chi_+ | \nabla \chi_+ \rangle = i\,\frac{\sin^2\theta/2}{r\sin\theta}\hat{\phi}$$

Next we want to evaluate

$\vec{\nabla} \times \langle \chi_+ | \vec{\nabla} \chi_+ \rangle$ only has a $\hat{\phi}$ component

Now in general we have

$$\vec{\nabla} \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta} A_\phi \sin\theta - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r A_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$\vec{\nabla} \times \langle \chi_+ | \vec{\nabla} \chi_+ \rangle = \vec{\nabla} \times \underbrace{\left\{ i \frac{\sin^2 \theta/2}{r \sin\theta} \right\}}_{A_\phi} \hat{\phi}$$

$$= \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left[\frac{i \sin^2 \theta/2}{r \sin\theta} \cdot \sin\theta \right] \hat{r} + \frac{1}{r} \left(-\frac{\partial}{\partial r} \left\{ \frac{r\, i \sin^2 \frac{\theta}{2}}{r \sin\theta} \right\} \right)$$

$$= \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left[\frac{i}{r} \sin^2 \theta/2 \right] \hat{r} + \frac{1}{r} \underbrace{\left(-\frac{\partial}{\partial r} \, i \frac{\sin^2 \theta/2}{\sin\theta} \right)}_{=0} \hat{\theta}$$

(no r dependence)

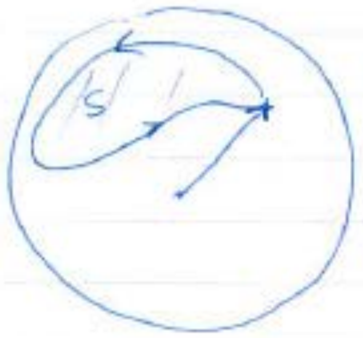
$$= \frac{1}{r\sin\theta} \frac{i}{r} \cancel{2} \sin\theta/2 \cos\theta/2 \cdot \frac{1}{\cancel{2}} \hat{r}$$

$$= \frac{i}{2r^2} \hat{r}$$

$$\gamma_+(T) = i \int_S \vec{\nabla} \times \langle \chi_+ | \vec{\nabla} \chi_+ \rangle \cdot d\vec{a}$$

$$= -\frac{1}{2} \int_S \frac{1}{r^2} \hat{r} \cdot d\vec{a}$$

integral is over an area on the sphere swept out by $\vec{B}$ in 1 cycle.



$$d\vec{a} = r^2 \, d\Omega \, \hat{r}$$

$$\gamma_+(T) = -\frac{1}{2} \int d\Omega = -\frac{1}{2} \Omega$$

Ω = solid angle subtended by the surface S at the origin.

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