POEM: Physics of Emergent Materials

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L1: Spin Orbit Coupling
L2: Topology and Topological Insulators

Tutorials: May 24, 25 (2017)
Scope of Lectures and Anchor Points:

1. Spin-Orbit Interaction
   - atomic SOC
   - band SOC: dresselhaus and rashba
   - symmetries: time reversal, inversion, mirror

2. Berry Phase and Topological Invariant
   - two level system
   - graphene

3. Hall effects
   - integer qhe and chern #
Recap
? \vec{d}(\vec{k})
\[ \text{NO SOC :} \]
\[
\mathcal{H} = \sum_{k \sigma} \varepsilon(k) C_{k \sigma}^+ C_{k \sigma}
\]
\[
= \sum_{k} \left( C_{k \uparrow} C_{k \downarrow}^+ \right) \left( \begin{array}{cc} \varepsilon(k) & 0 \\ 0 & \varepsilon(k) \end{array} \right) \left( \begin{array}{c} C_{k \uparrow}^+ \\ C_{k \downarrow} \end{array} \right)
\]

\[ \text{SOC \neq 0} \]
\[
\mathcal{H} = \sum_{k \sigma} \left( C_{k \uparrow} C_{k \downarrow}^+ \right) \left( \begin{array}{cc} h_{\uparrow \uparrow}^{(k)} & h_{\uparrow \downarrow}^{(k)} \\ h_{\downarrow \uparrow}^{(k)} & h_{\downarrow \downarrow}^{(k)} \end{array} \right) \left( \begin{array}{c} C_{k \uparrow}^+ \\ C_{k \downarrow} \end{array} \right)
\]
\[ \mathcal{H} = \sum_{\vec{r}} \left( C_{\uparrow} \downarrow \right) \begin{pmatrix} h_{\uparrow \uparrow}(\vec{r}) & h_{\uparrow \downarrow}(\vec{r}) \\ h_{\downarrow \uparrow}(\vec{r}) & h_{\downarrow \downarrow}(\vec{r}) \end{pmatrix} \begin{pmatrix} C_{\uparrow} \\ C_{\downarrow} \end{pmatrix} \]

Effect of SOC:
In real space as an electron hops from one site to another its spin can flip.

Using Pauli matrices
\[ \vec{h}(\vec{r}) = \epsilon(\vec{r}) \mathbf{I} + \vec{d}(\vec{r}) \cdot \vec{\sigma} \]
\[ = \epsilon(\vec{r}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + d_x(\vec{r}) \sigma_x + d_y(\vec{r}) \sigma_y + d_z(\vec{r}) \sigma_z \]

(suppress \( \vec{r} \))
\[ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + d_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + d_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
\[ = \begin{pmatrix} \epsilon + d_z & d_x - id_y \\ d_x + id_y & \epsilon - d_z \end{pmatrix} \]
Each eigenstate is doubly degenerate: Kramer’s degeneracy
Degeneracy between up and down spins is lifted by SOC except at specific TRIM: time reversal invariant momenta e.g. $p=0$.
Apply small Zeeman field along $\sim$

Gap opens up at the degenerate point.
Traverse the BZ always remaining in the lowest band. (adiabatic evolution).

$\rightarrow$ The electron spin will twist from $\downarrow$ for $-p$ to $\uparrow$ for $+p$
\[
\begin{align*}
  h(k) &= -\cos(k)\sigma_0 - \lambda \sin(k)\sigma_z - B\sigma_x = -\cos(k)\sigma_0 + \vec{d}(k).\vec{\sigma} \\
  d_x(k) &= -B \\
  d_z(k) &= -\lambda \sin(k)
\end{align*}
\]

\(B = 0, \lambda = 0\) vs. \(B = 0, \lambda = 1\)

Eigenvector corresponding to lower energy

Eigenvector corresponding to upper energy
$B = 0.1, \lambda = 1$

No Winding: hence topologically trivial
Why is all this important?
Interesting?
Useful?
Fundamentally important?
Game changer?
Can we get a device out of this?
...much more than a device!
A whole new paradigm for information storage that is topologically protected!
Hall Effect

\[ eE_y = ev_x B \]

\[ j_x = nev_x \]

\[ \rho_{xx} = \frac{E_x}{j_x} \]

\[ \rho_{xy} = \frac{E_y}{j_x} \]

Quantum Hall Effect

\[ \rho_{xy} = \frac{h}{e^2} \times \frac{1}{\nu} \]

\( \nu = 1, 2, 3, \ldots \)

filling factor
\[ \langle \sigma \rangle = [\langle \rho \rangle]^{-1} \]

\[ \sigma_{xx} = 0 \]

\[ \sigma_{xy} = \frac{e^2}{h} n \]

\[ \begin{align*}
\sigma_{xx} & \propto \rho_{xx} \\
\sigma_{xy} & = \frac{\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2} \\
\sigma_{xy} & = \frac{1}{\rho_{xy}}
\end{align*} \]
Topology: Shapes and invariants

quantum hall effect

precise quantization of Hall conductance to one part in billion even though system is disordered

\[ \sigma_{xy} = \frac{e^2}{h} n \]

Chern number

SOC is a source of “magnetic field” in momentum space k

What kinds of Hall effects can it produce?

What kinds of invariants does it lead to?
How large is the effective magnetic field arising from SOC?

\[ \mu_B \approx 10^{-4} \frac{\text{eV}}{\text{Tesla}} \]

\[ \lambda_{\text{soc}} \approx 0.1 \text{eV} \]

\[ \Rightarrow b_{\text{soc}}^{\text{eff}} \approx 1000 \text{Tesla} \]
insulators
Ordinary insulator (atomic or band)

Hall insulator
What is a topological insulator?

Bulk insulator
Conducting edge or surface states
Does not break TR
Hall Effects

Classical HE

Integer Quantum HE

Bulk-Boundary correspondence

Chern #=Z

Topological HE (r-space)

Anomalous HE (k-space)

Quantum Anomalous HE

Spin Hall Effect

Quantum Spin HE

SOC

Z₂
Topology and Topological Insulators

- Topology and Invariants: Gauss Bonnet formula
- Dynamic (usual) vs Geometric Phase
- Berry phase (topological invariant), can be measured
- Chern number and Quantum Hall Effect
- Degenerate bands, Berry Monopoles and Chern Number
- Spin Hall Effect
Topology and Topological Insulators

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Toys: Playdoh

cup \sim \text{donut}
Topology and geometry: Gauss-Bonnet formula

• Topological (quantized) numbers can be written as integrals of local quantities

• Gauss-Bonnet theorem. The Gaussian curvature $K$ of a 2D surface $M$ of genus $g$ integrated over the surface gives the Euler characteristic $\chi = 2 - 2g$

$$2 - 2g = \frac{1}{2\pi} \int_M K \, dA$$

• Can locally change the curvature, but the integral is quantized

• Relate such integrals to quantized Hall effects
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Topology and band theory

- Bloch theorem: states are labeled by a crystal momentum $\mathbf{k}$:

  $$\psi_{nk}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} u_{nk}(\mathbf{r})$$

- $u_{nk}(\mathbf{r})$ are lattice-periodic and are eigenstates of the Bloch Hamiltonian $H(\mathbf{k})$:

  $$H(\mathbf{k})|u_{nk}\rangle = E_n(\mathbf{k})|u_{nk}\rangle, \quad H(\mathbf{k}) = e^{-i\mathbf{k} \cdot \mathbf{r}} H e^{i\mathbf{k} \cdot \mathbf{r}}$$

  $n$ labels bands. Fully filled bands are separated by a gap from empty bands.

- Lattice symmetry implies periodicity in the reciprocal (momentum) space:

  $$H(\mathbf{k} + \mathbf{G}) = H(\mathbf{k}) \quad \Rightarrow \quad \mathbf{k} + \mathbf{G} \equiv \mathbf{k}$$

- Crystal momenta lie in a periodic Brillouin zone

  $$\mathcal{BZ} \cong \mathbb{T}^d$$

gruzberg
Main question: Consider an electron spin in a magnetic field:

\[ H = \vec{\sigma} \cdot \vec{B} \]

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]
\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Pauli Matrices

Suppose the magnetic field keeps the same magnitude but changes its direction.

At each time \( t \) the electron spin aligns itself to the local “up” direction.

After time \( t = T \) the \( \vec{B}(t) \) returns to its original value. What is the net phase picked up by the electron \( \gamma = 0 \)?
Berry phase

Phase acquired by wave function as the Hamiltonian is changed adiabatically and remains on the same eigenindex.

\[ H(\alpha) \rightarrow \text{set of parameters } \{\alpha_1, \ldots, \alpha_m\} \in M \]

\[ |n(\alpha)\rangle \text{ is the } n^{th} \text{ eigenstate for a set of parameters } \alpha \]

\[ H(\alpha)|n(\alpha)\rangle = E_n(\alpha)|n(\alpha)\rangle \]

\[ |\Psi_n(k)\rangle \]

Consider the evolution of the state \( |n(\alpha)\rangle \) as the parameters \( \alpha(t) \) evolve in time.
Adiabatic time scale for evolution of $\alpha(t)$ is slow compared to energy gaps. $|n(\alpha(t))\rangle$ remains an eigenstate but the wavefunction acquires a phase

$$|\psi_n(t)\rangle = e^{i\Theta_n(t)} e^{i\gamma_n(t)} |n(\alpha(t))\rangle$$
Dynamic "usual" phase:

$$\Theta_n(t) = -\frac{1}{\hbar} \int_0^t E_n(\alpha(t')) dt'$$

Geometric phase:

$$\gamma_n(t) = i \int_0^t \langle n(\alpha(t')) \left| \frac{d}{dt'} \right| n(\alpha(t')) \rangle dt'$$

(real)

* assume state is non-degenerate

For derivation: see NT notes
Also: Ballentine
"Quantum Mechanics"
Topology and Topological Insulators

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Berry connection

overlap of two wavefunctions infinitesimally separated in $\alpha$-space

$$\langle n(\alpha) \mid n(\alpha + \Delta \alpha) \rangle = 1 + \Delta \alpha \langle n(\alpha) \mid \nabla \mid n(\alpha) \rangle$$

$$\approx -i \Delta \alpha \cdot \vec{a}_n(\alpha)$$

vector potential

$$\vec{a}_n(\alpha) = i \langle n(\alpha) \mid \nabla \mid n(\alpha) \rangle$$
Specific case: Band structure of crystals

\[ \mathbf{\vec{\kappa}} = \mathbf{\vec{k}} \] crystal momentum

\[ E_n(\vec{\kappa}) \]

Bloch wave function:

\[ \psi_{n\mathbf{\vec{k}}}(\mathbf{\vec{r}}) = e^{i \mathbf{\vec{k}} \cdot \mathbf{\vec{r}}} \overline{u}_{n\mathbf{\vec{k}}}(\mathbf{\vec{r}}) \]

band index

periodic part

As \( \vec{\kappa} \) changes we map out an energy band set of all bands \( \rightarrow \) band structure

Brillouin zone will play the role of surface over which integrals will be calculated.
As \( \mathbf{K} \) is changed over the Brillouin zone, the phase picked up by the electron is

\[
\gamma_n = \oint \mathbf{a}_n \cdot d\mathbf{K} = \int \mathbf{n}_n \cdot d\mathbf{s}
\]

= total Berry flux within a Brillouin zone

\[
\mathbf{a}_{\mu}(\mathbf{K}) = i \langle u_{nK} \mid \frac{\partial}{\partial K_{\mu}} \mid u_{nK} \rangle
\]

along \( \mu \) direction

\[
\mathbf{a}_n(\mathbf{K}) = i \langle n\mathbf{K} \mid \nabla_{\mathbf{K}} \mid n\mathbf{K} \rangle
\]

• Berry phase is gauge invariant and is measurable.

[First derived by Pancharatnam (1955) in optics]

Berry Flux

\[
\mathbf{\Omega}_n = \nabla \times \mathbf{a}_n
\]

Chern # \( C_n = \frac{\gamma_n}{2\pi} \)

\[
N = \sum_{n=1}^{\# \text{occ bands}} C_n
\]

\[
\sigma_{xy} = \left( \frac{e^2}{\hbar} \right) N
\]
<table>
<thead>
<tr>
<th>Quantity</th>
<th>AB</th>
<th>Berry</th>
</tr>
</thead>
<tbody>
<tr>
<td>vector potential</td>
<td>$\mathbf{A}(\mathbf{r})$</td>
<td>$\hat{A}_n(\mathbf{k}) = \langle u_n^k</td>
</tr>
<tr>
<td>magnetic field</td>
<td>$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$</td>
<td>$\mathbf{\Omega}_n(\mathbf{k}) = \nabla_k \times \mathbf{A}_n(\mathbf{k})$ [Berry curvature]</td>
</tr>
<tr>
<td>flux</td>
<td>$\Phi = \oint_L \mathbf{A} \cdot d\mathbf{l}$</td>
<td>$\Phi_n = \int_{L} \hat{A}_n(\mathbf{k}) \cdot d\mathbf{k}$ [Berry flux]</td>
</tr>
<tr>
<td>integrated phase</td>
<td>$\gamma = 2\pi \Phi / \Phi_0$</td>
<td>$\gamma_n = 2\pi \Phi_n / \Phi_0$ [Berry phase]</td>
</tr>
<tr>
<td></td>
<td>$\Phi_0 = h/e$</td>
<td>$C_n = \Phi_n / \Phi_0$ [Chern #]</td>
</tr>
</tbody>
</table>
Degeneracies and two level systems

Berry curvature $\Omega_n(\vec{\mathbf{R}})$ is large near degeneracy points $\vec{\mathbf{R}}_0$ where energy levels cross:

$$E_n(\vec{\mathbf{R}}_0) = E_m(\vec{\mathbf{R}}_0)$$

genetically when two levels cross denote a

$$E_+(\vec{\mathbf{R}}) \geq E_-(\vec{\mathbf{R}})$$

Expand

$$H(\vec{\mathbf{R}}) \approx H(\vec{\mathbf{R}}_0) + (\vec{\mathbf{R}} - \vec{\mathbf{R}}_0) \cdot \nabla H(\vec{\mathbf{R}})$$

$$\overrightarrow{\Omega}_+(\vec{\mathbf{R}}) = \frac{i}{(E_+(\vec{\mathbf{R}}) - E_-(\vec{\mathbf{R}}))^2} \langle +, k | \nabla H(k_0) | -, k \rangle \times \langle -, k | \nabla H(k_0) | +, k \rangle$$

$$\overrightarrow{\Omega}_-(\vec{\mathbf{R}}) = -\overrightarrow{\Omega}_+(\vec{\mathbf{R}})$$
• Berry curvature:

\[ \vec{\Omega}_+ (\vec{k}) = i \frac{\langle +, \vec{k} | \sigma^- | -, \vec{k} \rangle \times \langle -, \vec{k} | \sigma^+ | +, \vec{k} \rangle}{4 k^2} \]

Choose \( \hat{z} \) along \( \vec{k} \)
then

\[ \sigma_z | \pm \rangle = \pm | \pm \rangle \]
\[ \sigma_x | \pm \rangle = | \mp \rangle \]
\[ \sigma_y | \pm \rangle = \pm i | \mp \rangle \]
\[ \sigma_y |\pm\rangle = \pm i |\pm\rangle \]

\[ \Omega_{+,x} = \Omega_{+,y} = 0 \]  
(proportional to \( \langle - | \sigma_z |+\rangle = 0 \))

\[ \Omega_{+,z} = i \frac{\langle + | \sigma_x | - \rangle \langle - | \sigma_y |+\rangle - \langle + | \sigma_y | - \rangle \langle - | \sigma_x |+\rangle}{4k^2} \]

\[ \Omega_{+,z} = -\frac{1}{2k^2} \]

\[ \vec{\Omega}_+ = -\frac{\vec{k}}{2k^3}, \quad \Omega_- = \frac{\vec{k}}{2k^3} \]

Monopole with charge \( \pm \frac{1}{2} \) at degeneracy point.
Berry curvature is large near the degeneracy points

Berry curvature = Monopole

\[ \Omega_+ = -\frac{k}{2k^3}, \quad \Omega_- = \frac{k}{2k^3} \]

Monopole with charge \( \pm \frac{1}{2} \) at degeneracy point.
If we integrate the Berry curvature over a sphere containing the monopole we get $2\pi$.

\[ \gamma_{\pm}(e) = \int_{S} \vec{\Omega}_{\pm} \cdot d\vec{k} = \frac{1}{2} \int_{S} \frac{\vec{E}^2}{k^2} \cdot d\vec{k} = \frac{1}{2} \text{ (solid angle subtended)} \]

\[ \text{Chern \# } \quad c_n = \frac{\gamma_n}{2\pi} \]
d vector to Berry Curvature

For a generic 2-level system
Berry flux (or Berry curvature)

\[ \vec{\Omega}_{t, jk} = \pm \frac{1}{2} \hat{d} \cdot (\partial_i \hat{a} \times \partial_k \hat{a}^\dagger) \]

= solid angle on unit sphere \( \hat{d} \)

\( \hat{d}(\mathbb{R}^2) \) maps the manifold \( M \rightarrow S^2 \)
e.g. \( T^2 \) (torus)
Berry curvature (analog of local magnetic field) on the lower band shows high density at the originally degenerate points. The mass terms have opened a gap at these points and created a monopole. However, you see that the Berry curvature at the K and K' points have opposite sign so if you add up the total Berry curvature or total magnetic field on this band adds up to zero.
Honey comb with sublattice potential

If you now map the d-vectors on the sphere they are essentially fluctuating around the north pole so the map can be continuously deformed to a point— the north pole— essentially the d-vectors can be combed straight up.
Berry curvature on the lower band shows high density at the originally degenerate points. The mass terms have opened a gap at these points and created a monopole. For the TR breaking case the Berry curvature at the K and K’ points have the same sign so if you add up the total Berry curvature or total magnetic field on this band adds up to 1.
Honey comb with time reversal breaking

The $d$ vectors around $K$ map to the north pole, the $d$ vectors around $K'$ map to the south pole. The entire sphere is covered and that gives a Chern number of 1.

$$\hat{d}(k) \in S^2$$

$$C_- = 1$$
Now include spin $\rightarrow$ spin hall effect:

spin $\uparrow$

spin $\downarrow$
Can the Berry phase be measured? Directly?
Berry Curvature in Graphene
Ultracold $^{87}$Rb in graphene-like honeycomb lattice

Three linearly polarized blue-detuned running waves at $120^\circ$ at $\omega_L$

Lattice acceleration from $\Delta\omega$ provides spin-independent force

$B$-field gradient creates spin-dependent force

(i) Resonant $\pi/2$-pulse creates coherent superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$ states

(ii) $B$-field gradient and lattice acceleration move atoms adiabatically along spin-dependent paths

(iii) $\pi$-pulse swaps the $|\uparrow\rangle$ and $|\downarrow\rangle$ states

(iv) Each cloud experiences opposite force from $B$-field gradient in the $x$-direction

(v) Second $\pi/2$-pulse closes interferometer and converts phase information into spin populations $n_{\uparrow,\downarrow} \propto \cos(\phi + \phi_{MW})$

Classical HE

Integer Quantum HE

Topological HE (r-space)

Anomalous HE (k-space)

Spin Hall Effect

Quantum Anomalous HE

Quantum Spin HE

Chern \# = Z

Bulk-Boundary correspondence

$\mathbb{Z}_2$
**QH vs. QSH**

**Quantum Hall**
- Breaks TR Symmetry
- Charge Conductivity Quantized: \( \sigma_H^C = \sigma_H^\uparrow + \sigma_H^\downarrow \)
- Magnetic Field
- Topological Invariant: Chern Number \( \in \mathbb{Z} \)

**Quantum Spin Hall**
- TR Symmetry Required
- Spin Conductivity Quantized: \( \sigma_H^S = \sigma_H^\uparrow - \sigma_H^\downarrow \)
- Spin-Orbit Coupling
- Topological Invariant: Even/Odd \( \in \mathbb{Z}_2 \)
Topological Insulator

- if two inequivalent insulators are in contact with each other, the gap must vanish at the boundary.

- Gapless states must exist at the boundary between inequivalent insulators

- The gapless states can also be classified topologically using the bulk-boundary correspondence

- Topological Invariant: $Z_2$ index (odd/even)
Trivial Insulator: $Z_2=$even

Topological Insulator: $Z_2=$odd

Topological invariant is $Z_2$:
Either 0 or 1
Calculated as product of parity eigenvalues at TRIM
Time reversal invariant momenta
(for systems with P symmetry)

On boundary the $Z_2$ index corresponds to the numbers of pairs of edge modes

Kramer’s Theorem → Degenerate Pairs at TR invariant momenta
….sheer poetry

Idea → prediction → discovery → precise quantization
Topology and Topological Insulators

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**2 level system**

*in a magnetic field*

\[ H = \vec{\sigma} \cdot \vec{B} \]

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]

\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Pauli matrices

\[ \vec{\sigma} \cdot \vec{B} = \begin{pmatrix} B_x & B_y - i B_z \\ B_x + i B_y & -B_z \end{pmatrix} = \begin{pmatrix} B_x & B_y \\ B_z & -B_z \end{pmatrix} \]

**Eigenspectrum:**

\[ \lambda_1 = +B \quad \chi_+ = \begin{pmatrix} \cos \theta/2 \\ \sin \theta/2 \end{pmatrix} \begin{pmatrix} e^{i\phi} \end{pmatrix} \]

\[ \lambda_2 = -B \quad \chi_- = \begin{pmatrix} \sin \theta/2 \quad -e^{i\phi} \end{pmatrix} \begin{pmatrix} \cos \theta/2 \end{pmatrix} \]

**Spinors**

Suppose the magnetic field keeps the same magnitude but changes its direction.

\[ \theta = 0 \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_- = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \]

\[ \theta = \pi/2, \phi = 0 \quad \chi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \chi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

At each time \( t \) the electron spin aligns itself to the local "up" direction.
After time $t = T$ the $\hat{B}(t)$ returns to its original value.

What is the net phase picked up by the electron $\gamma = 0$?

Overlap of wave function at two nearby times:

$$\langle \chi_{+}(\theta, \phi) \mid \chi_{+}(\theta + \Delta\theta, \phi + \Delta\phi) \rangle$$

$$= \frac{i \sin^2 \theta/2}{\sqrt{B \sin \theta}}$$

$$\alpha_{\phi} = \langle \chi_{+} \mid \hat{\nabla} \chi_{+} \rangle = \frac{i}{\sqrt{2}} \frac{\sin^2 \theta/2}{\sqrt{B \sin \theta}}$$

$$\vec{\Omega} = \vec{\nabla} \times \vec{\alpha} = \frac{i}{2} \frac{\sin^2 \theta/2}{\sqrt{2 \sin \theta}} \hat{r}$$

$$\gamma_{+}(T) = \int_{S} \frac{\vec{\Omega}, d\vec{s}}{r^2 \omega} \hat{r}$$

$$= -\frac{\Omega}{2}$$

$\Omega$ = solid angle subtended by the changing magnetic field over 1 time period.
To evaluate the Berry phase construct:

\[ \nabla \chi_+ = \frac{\partial \chi_+}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \chi_+}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial \chi_+}{\partial \phi} \hat{\phi} \]

\[ \nabla \chi_+ = \frac{1}{r} \left( \begin{array}{c} -\frac{1}{2} \sin \theta/2 \\ \frac{1}{2} e^{i \phi} \cos \theta/2 \end{array} \right) \hat{\theta} + \frac{1}{r \sin \theta} \left( \begin{array}{c} 0 \\ i e^{i \phi} \sin \theta/2 \end{array} \right) \hat{\phi} \]

\[ \langle \chi_+ | \nabla \chi_+ \rangle = \frac{1}{2 \pi} \left[ -\sin \theta/2 \cos \theta/2 \hat{\theta} + \sin \theta/2 \cos \theta/2 \hat{\theta} \\
+ 2 i \frac{\sin^2 \theta/2}{\sin \theta} \hat{\phi} \right] \]

\[ \langle \chi_+ | \nabla \chi_+ \rangle = i \frac{\sin^2 \theta/2}{r \sin \theta} \hat{\phi} \]
Next we want to evaluate
\[ \nabla \times \langle \chi_+ | \nabla \chi_+ \rangle \]
only has a \( \hat{\phi} \) component

Now in general we have
\[
\nabla \times \vec{A} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} A_\phi \sin \theta - \frac{\partial A_\theta}{\partial \phi} \right] \hat{r} \\
+ \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right] \hat{\theta} \\
+ \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi}
\]
\[ \nabla \times \langle \chi_+ | \nabla \chi_+ \rangle = \nabla \times \left\{ i \frac{\sin^2 \theta/2}{r \sin \theta} \right\} \phi \]

\[ = \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left[ \frac{i \sin^2 \theta/2}{r \sin \theta} \sin \theta \right] \right) \hat{r} + \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{3}{2} \frac{r \sin^2 \theta}{r \sin \theta} \right) \hat{\theta} \]

\[ = \left( \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left[ \frac{i \sin^2 \theta/2}{r} \sin \theta \right] \right) \hat{r} + \frac{1}{r} \left( -\frac{1}{\sin \theta} \frac{r \sin^2 \theta/2}{r \sin \theta} \right) \hat{\theta} \]

\[ = \frac{1}{r \sin \theta} \frac{i}{r} \frac{\partial}{\partial \theta} \sin \theta/2 \cos \theta/2 \hat{r} \]

\[ = \frac{i}{2r^2} \hat{r} \]
\[
\gamma_+(\tau) = i \int_S \nabla \times \left< \chi_+ | \nabla \chi_+ \right> \cdot d\vec{a}
\]

\[
= -\frac{1}{2} \int_S \frac{1}{r^2} \hat{r} \cdot d\vec{a}
\]

Integral is over an area on the sphere swept out by \(B\) in 1 cycle.

\[d\vec{a} = r^2 \, d\Omega \, \hat{r}\]

\[
\gamma_+ (\tau) = -\frac{1}{2} \int d\Omega = -\frac{1}{2} \Omega
\]

\(\Omega = \text{solid angle subtended by the surface } S \text{ at the origin.}\)
Topological and Topological Insulators

- Topology and Invariants: Gauss Bonnet formula
- Topology and Band Theory
- Dynamic (usual) vs Geometric Phase
- Berry phase (topological invariant), can be measured
- Chern number and Quantum Hall Effect
- Degenerate bands, Berry Monopoles and Chern Number
- Spin Hall Effect