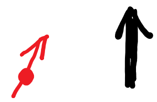
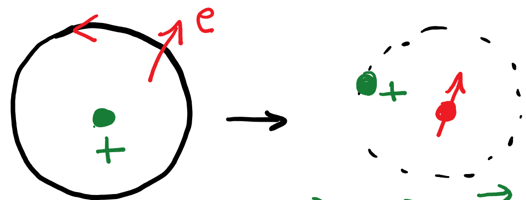
1. atomic SOC :

$$\mathcal{H}_{\text{Zeeman}} = \vec{\sigma} \cdot \vec{B}$$



$$= \lambda\, \vec{\sigma} \cdot \vec{L}$$



$$\vec{B} \sim \vec{v} \times \vec{E}$$
$$\propto \vec{L}$$

2. bands SOC :

$$\mathcal{H} = \lambda \underbrace{\vec{b}(\vec{k})}\cdot \vec{\sigma}$$

k-dependent magnetic field
preserves TR

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{b}(\vec{k}) \cdot \vec{\sigma} = b_x(\vec{k})\sigma_x + b_y(\vec{k})\sigma_y + b_z(\vec{k})\sigma_z$$

$$= \begin{pmatrix} b_z & b_x - i b_y \\ b_x + i b_y & -b_z \end{pmatrix}$$

At each $\vec{k}$ point $\vec{b}(\vec{k})$ changes direction
electron spin aligns with $\vec{b}$.

eigen-spectrum :

$$\varepsilon_+ = +|\vec{b}| \qquad \chi_+ = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi} \sin\theta/2 \end{pmatrix}$$

$$\varepsilon_+ = +|\vec{b}| \qquad \chi_+ = \left(e^{i\phi} \sin\theta/2 \right)$$

$$\varepsilon_- = -|\vec{b}| \qquad \chi_- = \begin{pmatrix} e^{-i\phi} \sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$$

3. Berry Phase

overlap of wavefunction spinor $\chi_{n=\pm}(\vec{k})$ at two nearby points in k

$$\langle \chi_n(\vec{k}) \,|\, \chi_n(\vec{k}+\vec{\Delta k}) \rangle \simeq e^{i\, \vec{\Delta k} \cdot \vec{a}_n(\vec{k})}$$

$$\underbrace{\vec{a}_n(\vec{k})}_{\text{Berry Connection}} = i \langle \chi_n(\vec{k}) | \vec{\nabla}_k | \chi_n(\vec{k}) \rangle$$

For Bloch states

$$\chi_n(\vec{k}) = u_n(\vec{k})$$

band index

$$\Omega_n(\vec{k}) = \vec{\nabla} \times a_n(\vec{k})$$

Berry Flux

$$\gamma_n = \int_{\substack{BZ \\ T^2 \\ (\text{torus})}} \vec{a}_n(\vec{k}) \cdot \vec{dk} = \int_S \vec{\Omega}_n(\vec{k}) \cdot \vec{dS}$$

Berry Phase

$$C_n = \frac{\gamma_n}{2\pi} \qquad C = \sum_{\substack{n \\ \text{occupied}}} C_n$$

Chern # for band n

$$\boxed{\sigma_{xy} = \frac{e^2}{h} C}$$

4. Topological invariant

$$\sigma_{xy} = \frac{e^2}{h} \frac{1}{2\pi} \int_{T^2} d^2k \ \Omega^z_n(\vec{k})$$

$$= \frac{e^2}{h} \frac{1}{4\pi} \underbrace{\int_{T^2} d^2k \ \hat{d} \cdot \left(\frac{\partial \hat{d}}{\partial k_x} \times \frac{\partial \hat{d}}{\partial k_y} \right)}_{\text{Solid angle subtended}}$$